Hedging Prepayment Risk under Equilibrium Pricing

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ABSTRACT

Due to the complex prepayment behavior, mortgage contracts and their derivatives are generally priced using Monte Carlo simulations. The typical approach used by the industry, which involves simulating interest rates under risk-neutral space and attaches with actuarially estimated prepayment rate function, results in internal inconsistency. This paper is the first to directly investigate the impact of such an error. Following Cox, Ingersoll, Ross (1985)'s general equilibrium setting, we use a market risk price measure to back out the real interest rate process from the observed yield curve. This allows us to model the mortgages using the consistent real measures of interest rate and the prepayment rate as a function of the underlying interest rate process.

The results show that while the mixed approach may yield reasonable estimate for a par mortgage, it tends to generate pricing biases for premium and discount MBSs. The more severe biases are found to be on the risk measurements. The duration can be biased by up to 20 percent. The bias can be on different direction depending on the particular components of an MBS. The convexity bias is even greater. These biases in risk estimates can introduce failure in hedging against interest rate and prepayment risks. Over-hedging can also lead to extra hedging cost.

Duration and Convexity of Mortgage-Backed Securities for Risk Averse Investors

1 INTRODUCTION

Mortgages are fixed-income instruments with embedded interest rate, prepayment and default risks. Because of the huge market volume, its pricing and risk analysis has been a popular research subject over the past two decades. The unique feature of an MBS compared with a straight bond is the uncertainty of the timing of the principal being returned. That is, the actual cash flows of an MBS are driven by the realized prepayment rates, which in turn are mainly driven by the realized market interest rate scenario. The prepayment behavior could be modeled by a ruthless option pricing approach (see Kau et al. (1992) for this line of approach) or an empirically estimated approach (see Calhoun and Deng (2002) for this line of approach).

Following Hall (1985), many researchers applied option pricing models to value mortgages and mortgage-backed securities with lattice or Monte Carlo approaches. The pure option approach literature assumes that a ruthless borrower would exercise these embedded options so as to minimize the value of the mortgage, which is a liability to the borrower.[1] However, empirical evidence showed that the prepayment and default options are usually not exercised optimally, at least at the finite risk factors models. The prepayment rates implied by the ruthless prepayment option models are vastly different from the market experience. Deng, Quigley, and Van Order (2000) refer the fact that most households do not optimally exercise the prepayment option, and call them "wood heads". Stanton (1995) and others attempt to explain the non-optimal exercise of the options by the presence of refinancing transaction costs. However, after a few refinement of the model, the implied prepayment cost is still far higher than reasonable. As a result, almost 20 years after the prepayment option theory introduced by Hall

[1] For example, see Kau et al. (1992), Hilliard, Kau, and Slawson (1998), Follain, Scott, and Yang (1992), Chen and Yang (1996).

(1985), this approach still remains at the academic research stage and has not been adopted by industry practitioners.

To more accurately capture the prepayment behavior, various econometric models are estimated to better reflect the market reality. As of today, the most widely applied approach in the industry has been to use historical data to empirically estimate the prepayment rates of MBSs as a function of their contract characteristics and market conditions, mainly the concurrent and previous market interest rates. Almost all MBS investors are using a version of empirically estimated prepayment rate model. The most sophisticated investors develop their own proprietary models based on public and internal data. The smaller players can easily purchase third-party models from Bloomberg, Andrew Davidson, AFT, and numerous others. Regulatory supervisors such as OTS, OCC, or OFHEO also have built their own versions from the data of the specific segment of the MBS market they regulate. We would like to emphasize that these empirically estimated prepayment functions all capture the relationship between the historically observed actual prepayment rates and actual interest rates.

Because of its interest rate derivative nature, MBSs are generally priced similar to all fixed income securities by applying risk-neutral term and volatility structural models derived from market Treasury curves.[2] Then Monte Carlo or lattice models are implemented to simulate possible future cash flows by combining the uncertain interest rate paths and the prepayment and functions. Specifically, the risk factors are simulated in the risk-neutralized space, while the prepayment/default functions are estimated based on actual realized outcomes.

Such an approach produces serious internal inconsistency. While the future interest rate distribution is measured under the risk-neutral probabilities, the prepayment and default functions are all linked to the real measures. The refinancing incentive is the single most important factor determining prepayment rate of a mortgage. Empirically, it is measured by comparing the mortgage contract rate with the concurrent market rate. When estimating the econometric functions, historical actual realized market rates are used. However, when this function is applied in future cash flow projection, the market rates are simulated in a distorted risk-neutral space rather than in the real space. Under a risk-averse environment, the expected rate in the risk-neutral space would be higher than that in the real space. This internal inconsistency could lead to under-estimate of the mortgage prepayment speed.

In this paper, we attempt to investigate the potential impact in pricing and risk measurement in MBSs caused by this model inconsistency. Specifically, we implement a single index asset pricing model under a hypothetical world where interest rate is the only risk factor. The implemented pricing model is built upon the "real space" as opposed to the "risk neutral space". The use of the real space enables the internally consistent integration of the real term structure model (instead of the usually applied risk-neutral term structure model) and the (real) default and prepayment models. With such a model, all risk factors, interest rates, prepayments, and defaults can be brought to a consistent pricing environment for the first time. For ease of demonstration purpose, the default risk is left out of this model. This is equivalent of assuming the MBSs are fully insured for default risks.

It is arguable that one can also achieve internal consistency by adopting the risk-neutral interest rate and the risk-neutralized prepayment (and default) functions. Such an approach is theoretically sound, but empirically difficult to achieve. The historical performances data of millions of mortgages contain valuable information and are the comparative advantage of the empirical approach over the ruthless prepay and default assumptions. The desire to take advantage of these data to more accurately project future prepayment and default behavior is always the highest priority for real world investors.

Equilibrium pricing becomes necessary when the underlying asset is not publicly traded. In their seminal paper, Cox, Ingersoll, and Ross (1985) demonstrate that the equilibrium pricing and risk neutral pricing both yield the identical result as long as *continuous time trading* is

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[2] For investors taking on default risk, a second stochastic process for the value of the underlying collateral property

permitted.[3] When continuous portfolio rebalance is feasible, the underlying asset and its contingent claims are perfect correlated and risk-free portfolio can be achieved. This allows us to price the contingent claims with the arbitrage-free models. When continuous trading is not possible, the perfect correlation between the underlying asset and its derivates breaks down. Perfect hedging under discrete trading is generally not possible. As a result, arbitrage free or risk-neutral pricing becomes impermissible[4] and equilibrium pricing is the only solution. The less liquidity and the discrete prepayment information arriving time of the MBS market make this constraint particular important.

The pricing model applied in this paper is expanded from the Cox-Ingersoll-Ross (1985a) general equilibrium pricing framework. Through a series of numerical examples, we demonstrate that large pricing biases can exist due to the internal inconsistency of industry common practice. Even more significant biases are found in the risk measurements.

The paper is organized as follows. Section 2 describe the MBS pricing model under a consistent real probability space. Section 3 provides numerical examples to demonstrate the application of the pricing model to simple MBSs and IO and PO strips. The impacts on investor's interest rate risk hedging strategy by the inconsistent treatment of the interest rate risk simulation and the prepayment function are summarized in Section 4. Finally the paper is concluded in Section 5.

2. THE CONSISTENT MODEL

The fundamental risk factor driving prepayment rates and the MBS value as a fixed-income asset is the market interest rate. Two approaches are available to model such fixed

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may be also adopted.

[3] The equilibrium pricing is System I, and risk neutral pricing is System II in Cox, Ingersoll, and Ross (1985).

[4] It is possible for the risk neutral pricing methodology to co-exist with discrete trading but the option must be written on the *only* asset in the economy – usually assumed to be the aggregated consumption. See Brennan (1979) for a detailed discussion of this point.

incomes: the risk-neutral approach and the risk-adjusted return approach. We assume the true instantaneous interest rate follows the Vasicek (1977) process:

(1) $$dr = \alpha(\mu - r)dt + \sigma dW^P$$

where α is the mean-reverting speed, μ is the long-term mean-reverting level of the interest rate, σ is its volatility, and W^P is the standard Brownian motion under the actuarial measure. Under the risk-neutral framework, after variable transformation, the process becomes:

(2) $$dr = \alpha(\mu^N - r)dt + \sigma dW^Q$$

where W^Q is the standard Brownian motion under the risk-neutral measure, and

(3) $$\mu^N = \mu - \frac{\sigma q}{\alpha},$$

where q is the market price of risk. The interest rate at time T in the real and risk-neutral spaces are both normally distributed with the same variance of

(4) $$Var_t[r(T)] = \frac{\sigma^2(1 - e^{-2\alpha(T-t)})}{2\alpha},$$

but with different mean values of:

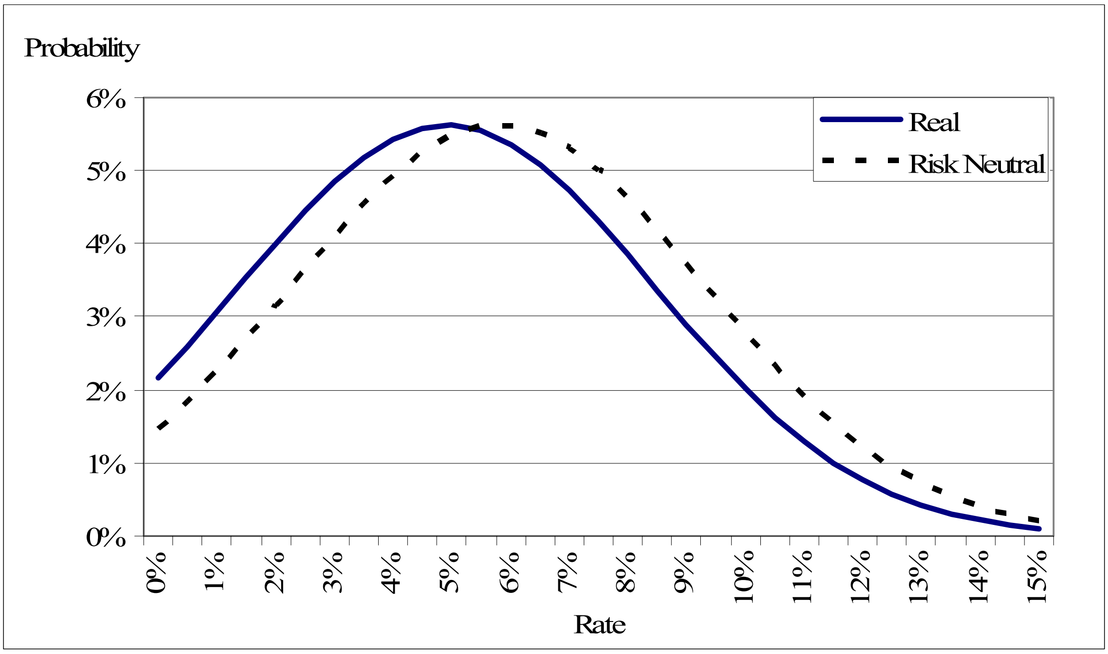
(5) $$E_t[r(T)] = r(t)e^{-\alpha(T-t)} + \mu\left(1 - e^{-\alpha(T-t)}\right), \text{ and}$$

(6) $$E_t^N[r(T)] = r(t)e^{-\alpha(T-t)} + \left(\mu - \frac{\sigma q}{\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right).$$

Note that when investors are risk averse ($q < 0$), $E_t[r(T)] < E_t^N[r(T)]$. Also, the risk neutral process can be viewed as a special case when the market risk price is zero, *i.e.*, $q = 0$.

Figure 1 plots the distribution of $r(1)$ with $r(0) = 5\%$, $\alpha = 0.25$, $\mu = 7\%$, $\sigma = 4\%$, and $q = -0.25$. The solid line represents the interest rate distribution under the real measure and the dotted line under the risk-neutral measure. Due to investors' risk aversion, the risk neutral process is more likely to obtain higher interest rates (density function higher for the higher rates) than that of the real process.

Figure 1: Real versus Risk Neutral Interest Rate Distributions



Zero Coupon Bond

Under the assumption of continuous discounting, the market value of a discount bond (*i.e.* $1 face value) can be found using the closed-form solution derived by Vasicek (1977) as:

(7) $$P(t,T) = e^{-r(t)F(t,T)-G(t,T)}$$

where

$$F(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}, \text{ and}$$

$$G(t,T) = \left(\mu - \frac{\sigma q}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right)\left(T - t - F(t,T)\right) + \frac{\sigma^2 F^2(t,T)}{4\alpha}.$$

This implies that the risk-free rate of return of an investment between time t and T can be computed as:

(8) $$r(t,T) = \frac{1}{P(t,T)} = e^{r(t)F(t,T)+G(t,T)}.$$

Similarly, the value of a zero-coupon bond with a longer maturity date ($\tau > T$) at time T can be computed as:

(9) $$\widetilde{P}(T,\tau) = e^{-\widetilde{r}(T)F(T,\tau)-G(T,\tau)},$$

where $\widetilde{r}(T)$ is the instantaneous rate at time T, which is a random variable that follows a normal distribution specified by equations (4) and (5). Hence, $\widetilde{P}(T)$ follows a lognormal distribution. The value of this bond at time t can be computed by a risk-neutral approach as equation (7) or alternatively by using a risk-adjusted discount rate approach as:

(10) $$P(t,\tau) = \begin{cases} E_t^Q\left(\exp\left(-\int_t^T r(u)du\right)\widetilde{P}(T,\tau)\right) = E_t^Q\left(\widetilde{P}(T,\tau)\right)e^{-\hat{r}(t,T)(T-t)} \\ e^{-k(t,T)(T-t)}E^P\left(\widetilde{P}(T,\tau)\right) \end{cases}$$

where $k(t,T)$ is the risk-adjusted rate of return for the holding period t to T at the market risk price q. For an investor that does not intend to trade between time t and T, there is risk involved in the return from holding the zero-coupon bond with maturity date τ longer than T. We now compare the expected return of this investment during the holding period under the real and risk-neutral measures:

(11) $$R(t,T) = \frac{E^P\left(\widetilde{P}(T,\tau)\right)}{P(t,\tau)} = e^{k(t,T)(T-t)} > e^{\hat{r}(t,T)(T-t)} = \frac{E^Q\left(\widetilde{P}(T,\tau)\right)}{P(t,\tau)}$$ when investors are risk averse ($q < 0$).[5]

In the single-risk-factor economy, the higher is the volatility of the payoff of the bond at the end of the holding period, the higher is the expected return. Also, the higher is the market risk price, the higher the expected return would be, holding everything else constant. Since there is only risk factor in this world, the returns among all zero coupon bonds with different maturity

[5] Note that $E^p(P) > E^Q(P)$ because from equation (3.5), $E^p(r_T) < E^Q(r_T)$ when $q < 0$.

dates are all identical. The variance of this single period return remain the same for both risk-neutral and the real probability spaces as:

(12)
$$
\begin{aligned}
\operatorname{var}\left[\frac{\widetilde{P}(T,\tau)}{P(t,\tau)}\right] &= \frac{1}{P(t,\tau)^2}\operatorname{var}[\widetilde{P}(T,\tau)] \\
&= e^{2r(t)F(t,\tau)+2G(t,\tau)}\operatorname{var}\left[e^{-\widetilde{r}(T)F(T,\tau)-G(T,\tau)}\right] \\
&= e^{2r(t)F(t,\tau)+2G(t,\tau)-2F(T,\tau)}\left(e^{-2E[r(T)]+2V[r(T)]}-e^{-2E[r(T)]+V[r(T)]}\right)
\end{aligned}
$$

Coupon Bonds

By the law of one price, a coupon bond can be derived as the sum of multiple zero-coupon bonds. That is:

(13) $$B(t,\tau)=\sum_{t+1}^{\tau}\pi_u P(t,u)$$

where π_u is the cash flow of the bond at time u.

Mortgage-Backed Security

An MBS is a fixed-income instrument with uncertain prepayment rates (we assume the default risk has been fully insured away). When the market interest rate drops, borrowers may prepay the existing mortgage by borrowing at a new lower rate. Due to such refinancing behavior, the prepayment rates of MBSs have been found to be highly negatively related to the market interest rate. However, the relationship is not linear but is asymmetric. The sensitivity has been found to be much higher during a declining rate environment than during a rising rate environment. MBS investors have devoted tremendous effort to estimate the prepayment rate function using historically observed prepayment rates. If prepayment rates can be accurately modeled, then one can simulate future interest rate movements and estimate the cash flows of an MBS. Typically, prepayment models are found to be path-dependent. That is, the current prepayment rate is dependent on the concurrent interest rate but also on the rate history from the MBS issuance date up to today. Due to this path-dependent feature and the complex functional form of the prepayment models, MBSs are generally priced using Monte Carlo simulation approaches.

Following equation (13), the cash flows of a MBS can be striped into zero coupon bonds at different payment dates with uncertain final payoffs. For example, a 30-year MBS can be stripped into 360 individual zero-coupon bonds, each entitling the buyer to receive the cash flow at one particular month during a 30-year period. Therefore, the MBS becomes a portfolio of zero-coupon bonds. The only additional complication is that the final payoff of each individual zero-coupon bond is uncertain. Following the rule of one price, the MBS value will be the same as the sum of the individual zero-coupon bonds:

$$(14) \qquad MBS(t,\tau) = \sum_{u=t+1}^{\tau} \frac{E^{Q}\left(\tilde{\pi}(u)\right)}{e^{r(t,u)(u-t)}} = \sum_{u=t+1}^{\tau} \frac{E^{P}\left(\tilde{\pi}(u)\right)}{e^{K(t,u)(u-t)}}$$

where $K(t,u)$ is the risk-adjusted rate of return between time t and time u for the given uncertainty involved in the cash flow at time u, $\tilde{\pi}(u)$.

With the availability of an empirically estimated prepayment function, there are two alternative approaches to applying equation (14) without introducing internal inconsistency. First, one can simulate interest rate paths using the actuarial probability measure with the actuarial-based prepayment function to generate cash flows, and then discount by the risk-adjusted return. The second solution is to simulate the risk-neutral interest rate paths and use a risk-neutralized prepayment function to estimate the cash flows, and then discount by the risk-free rates. The first approach requires the estimation of the risk-adjusted returns. The second approach requires the derivation of a risk-neutralized prepayment function.

In this paper, we present a solution using the first approach. While the second approach might be theoretically feasible, it is unclear whether there is enough information to correctly transform the prepayment function from the actuarial space to the risk-neutral space.

In order to price the MBS under the actuarial measure, we must estimate the risk-adjusted return for each single investment holding period, $K(t,u)$. Under the standard Sharpe-Lintner

Capital Asset Pricing Model framework, we can write the risk-adjusted return of a particular payment date strip as:

$$K(t,u) = \frac{1}{P(t,u)} + \beta(t,u)\left[R(t,u) - \frac{1}{P(t,u)}\right] \tag{15}$$

where $\beta(t,u)$ is the measure of systematic risk of the final payoff at time u against the "market portfolio", and $R(t,u)$ is the expected return of the market portfolio during the period from time t to u. Without lose of generality, we choose a zero-coupon bond with extremely long maturity (100 years) as the market portfolio to capture the interest rate risk. As a result, $R(t,u)$, the market portfolio return can be computed using equation (11). $\beta(t,u)$ is the well-known systematic risk coefficient:

$$\beta(t,u) = \frac{\operatorname{cov}\left[\frac{\widetilde{P}(u,\tau)}{P(t,\tau)}, \frac{\widetilde{\pi}(u)}{\pi(u|t)}\right]}{\operatorname{var}\left[\frac{\widetilde{P}(u,\tau)}{P(t,\tau)}\right]} = \frac{P(t,\tau)}{\pi(u|t)} \frac{\operatorname{cov}\left[\widetilde{P}(u,\tau), \widetilde{\pi}(u)\right]}{\operatorname{var}\left[\widetilde{P}(u,\tau)\right]} \tag{16}$$

where $\pi(u|t)$ is the market value of the cash flow $\pi(u)$ at time t. $\beta(t,u)$ and $\pi(u|t)$ can be solved by iteration until convergence.

4. NUMERICAL EXAMPLE

Numerical results are generated to obtain numerical results to investigate the potential impacts of the inconsistency in risk-neutral interest rate space and an actuarial based prepayment function. To best capture the stochastic interest rate environment and the path dependent nature of MBSs, Monte Carlo simulations are applied to implement the model introduced in Section 2. The implementation includes the following steps:

a. Simulate 500 paths of instantaneous interest rates up to the MBS maturity date (*e.g.*, 360 months) by equations (4) and (5).

b. Compute the market value of the "market portfolio" or the 100-year zero coupon bond using equation (7). With the 500 paths, we will compute 500 possible prices of the 100-year zero coupon bond prices at each point in time in the next 360 months.

c. Compute the expected return of the market portfolio by equation (11). The average return is the average of the 500 market values of the 100-year zero coupon bond at time t divided by the value of the same bond at time zero.

d. Compute the risk-free return for each time horizon by equation (8). This is the return on a zero coupon bond with maturity equal to the specific time period.

e. Simulate cash flows of the MBS for each point in time. This is done by combining the amortization schedule, the conditional prepayment rate function, and the specific path of short term interest rates.

f. Compute the expected cash flow, $E_0(CF_t)$, of the MBS at each payment date. This is taking the simple average of the cash flow occurred at time *t* among the 500 simulated interest rate paths.

g. Iteratively solve for the beta and market value at time zero of the cash flow at each payment date against the market value of the 100-year zero coupon bond by using equation (16).

h. The value of the MBS at time zero is the sum of the values of cash flows computed in step g.

Assumed Prepayment Function

The uncertainty of future cash flows will vary with the specific prepayment function being used. To provide a more precise sense of the direction and the magnitude of the biases caused by this mixed use of risk-neutral interest rate projections and real prepayment models, we assume a simple and flexible, but reasonable, prepayment function to provide numerical examples. The specific prepayment function we assume takes the following form:

(17) $$CPR_v = \left(\frac{r_c}{r_m}\right)^h PSA_v$$

where CPR_v is the conditional prepayment rate at age equal to month v, r_c is the contract mortgage rate, r_m is the market mortgage rate, h is the sensitivity of the *CPR* to the change of the market mortgage rate, and PSA_v is the 100% *PSA* prepayment speed at age v, which starts at 0.2% per annum at the first month one rise by 0.2% every month untill it reaches 6% per annum at the 30-th month, then remains constant untill the maturity date.

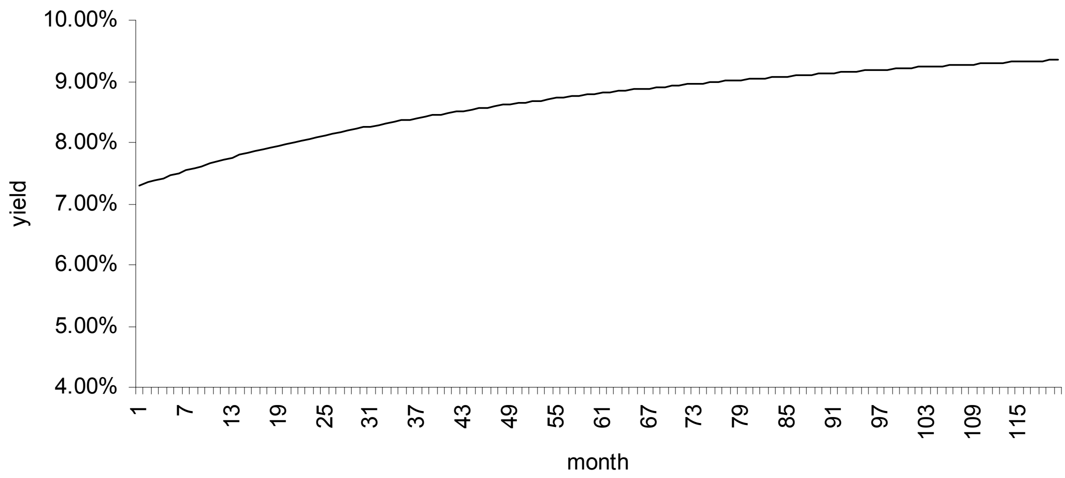
To be consistent with real life experience, we set the upper bound of the conditional prepayment rate to be 70% in the numerical example. Over the last 20 years, even when the market mortgage rate is far below the mortgage contract rate, the annualized conditional prepayment rates hardly ever exceed 70%. The remainder of the borrowers could be bounded by other financial constraints such as low market house value or low household income which prevents the borrower from qualifying for a new mortgage. It could also be due to the slowness of the borrower to act on the refinancing, or simply because they are "wood heads".

The example we use to illustrate the issue has the parameters of the interest process defined earlier. Additional parameters associated with the MBS contract are

r_c	10%
T	10 years
q	-0.25
R_0	7%

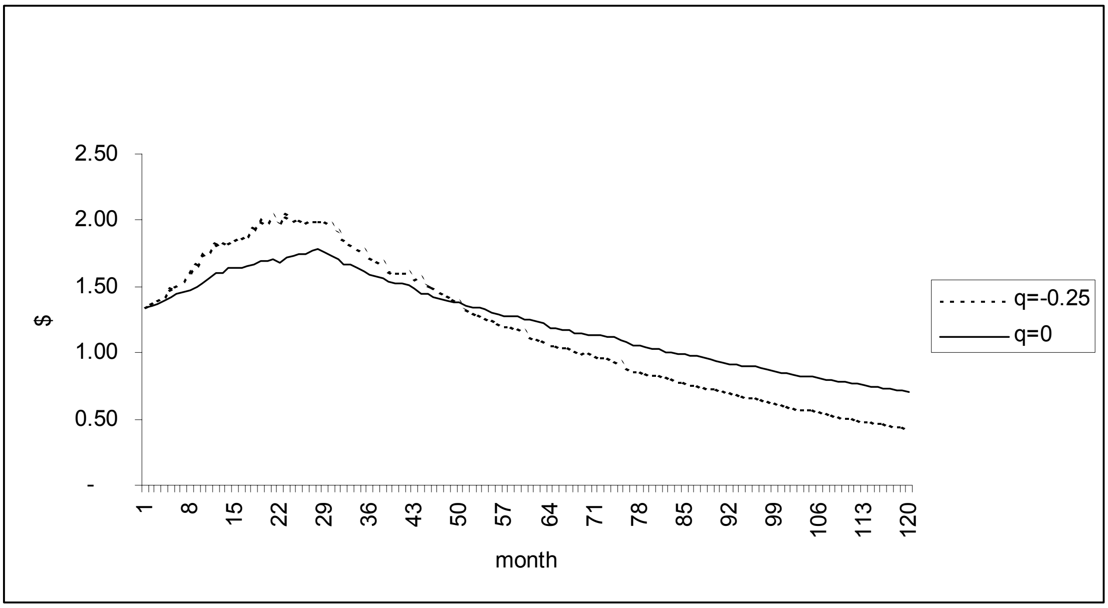
These parameters imply the shape of the default-free yield curve in the following graph:

Figure 2: Implied Yield Curve



To capture the path-dependent feature, we estimate the value of an MBS using Monte Carlo simulations. We first simulate the monthly default-free rates over the next 10 years by the following discrete approximation of the interest rate process used by the Vasicek model using equation (2) with a different market risk price. Then, we apply the assumed prepayment function to compute the principal amortization and prepayment. The same process is repeated 500 times. Shown below are the simulated principal and interest payments under the real and risk-neutral measures, for a simple MBS, and for interest only and principal only strips.

Figure 3: Expected Cash Flows: MBS



Expected cash flows are the average values for each simulated point in time of the 500 sample paths. The MBS expected cash flows over the next 120 months given the prepayment function assumed present a humped shape, under both risk-neutral ($q = 0$) and real ($q = -0.25$) measures. Yet the expected cash flow curve peaks at month 29 in the risk-neutral case while the under the real measure it peaks at month 21. Another noticeable difference is the difference between the two measures at the long end. The risk-neutral case has expected cash flows substantially higher than those in the real case.

Figure 4 presents the expected cash flows of the Interest Only (IO hereafter) strip under the two different measures. As we can see, IO strip present a very different cash flow profile from that of the MBS. The large differences between the two measures occur in the midlife of the contract, not at the end as in the MBS case. The difference disappears at the long end. Another interesting observation is that there is no crossover of the two expected cash flow curves between the two measures. The risk-neutral cash flows dominate the real cash flows for all the months. Finally, also note that due to the prepayment function assumed, the expected cash flows of the IO strip present a monotonic decreasing function over time.

Figure 4: Expected Cash Flows: Interest Only

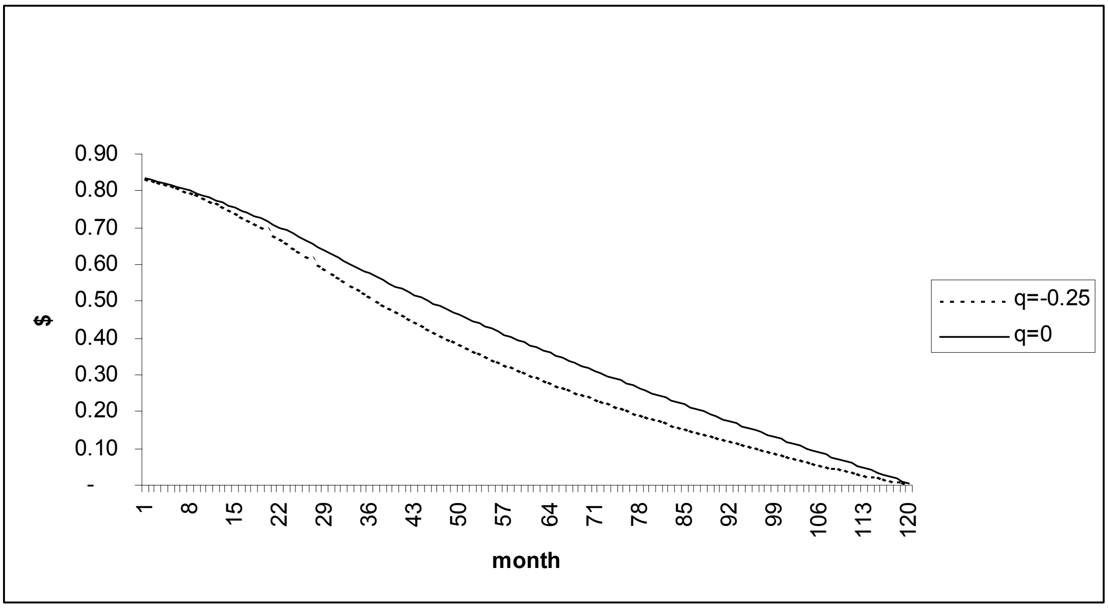
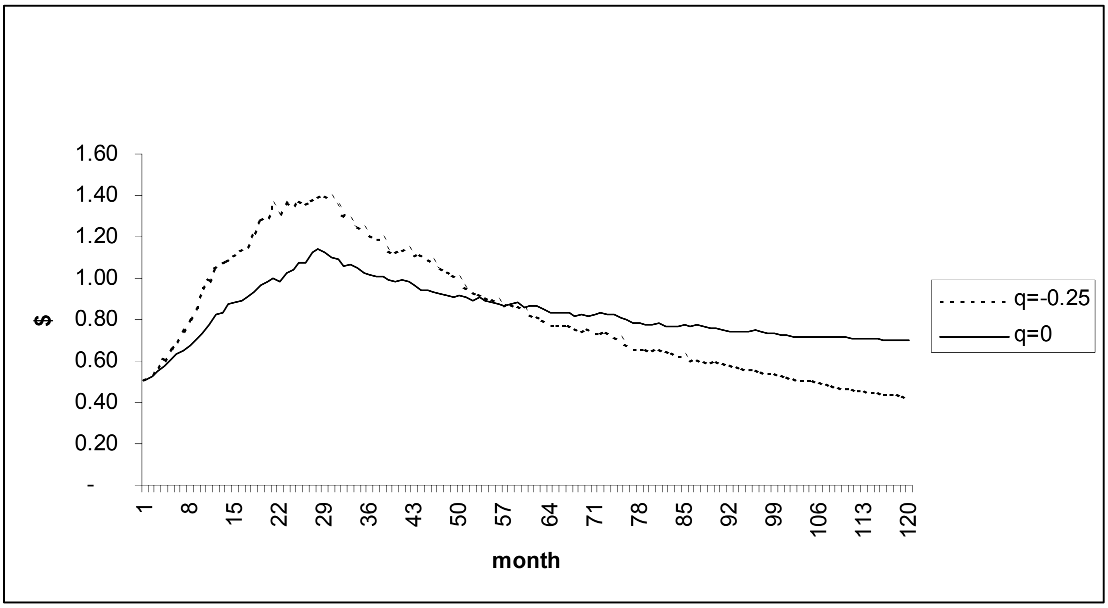


Figure 5: Expected Cash Flows: Principal Only



The expected cash flows of Principal Only (PO hereafter) are plotted in Figure 5. The PO expected cash flows present a similar pattern to those of the MBS. But the peak of the cash flow curve under the real measure is moved to month 30, which is about the same time as the peak under the risk-neutral measure. Clearly, this is because the IO curve under the real measure decreases over time faster than the IO curve under the risk-neutral measure.

In additional to the expected cash flows, we also present the present values of these expected cash flows using the capital asset pricing model described in a previous section. Note that large real vs. risk-neutral cash flow discrepancies will diminish because higher risk-neutral cash flows will be discounted at the risk-free rates (for cash flows at different times) that are higher than the discount rates under the real measure.

When we sum across present values at different times, we arrive the total present values of MBS, IO, and PO. The price discrepancies between the real measure where the market price of risk is –0.25 and the risk-neutral measure are 5%, 2%, and 7% for MBS, IO, and PO respectively. These differences may be small since many investors argue that the liquidity premiums in these markets are easily in the neighborhood of multiple percentages. However, as we demonstrate in the following table, the discrepancies will quickly become substantial as investors are slightly more risk averse. For example, for MBS, when the market price of risk increases from –0.25 to –0.5, the price difference increases from 5% to 12%. In the case of the IO, when the market price of risk increases from –0.5 to –1, the price difference increases from 5% to 17%.[6]

Table 1 Value of Securities

	Value		
q	MBS	IO	PO
0	107.67	36.23	71.45
-0.1	105.69	35.89	69.80
-0.25	102.10	35.46	66.64
-0.5	95.74	34.59	61.15
-1	89.07	31.00	58.07
	% Bias in price when assuming $q = 0$		
-0.1	2%	1%	2%
-0.25	5%	2%	7%
-0.5	12%	5%	17%
-1	21%	17%	23%

[6] It is not unusual for the MBS markets to present the market price of risk as high as –2.

5. INTEREST RATE RISK MANAGEMENT IMPLICATION

For large mortgage and MBS portfolio investors, the ability to actively manage the interest rate risk is as important as the ability to accurate price a deal. Many investors in the MBS market are taking the buy and hold strategy. For these investors, the actual cash flows generated from the security over time are much more important to the daily fluctuation of the MBS prices. Mortgage banks, whose portfolios are composed of IO equivalent servicing contracts, are subject to the most severe prepayment risks. With the illiquid mortgage servicing right market and the discrete information arrival time, the ability to accurately hedge the interest rate and prepayment risk is among the top priorities of these institutions.

In the following, we examine the interest rate sensitivity of the equilibrium model. We look at the two usual interest rate risk measures – duration and convexity, which are represented by the slope and curvature of the price as a function of the interest rate. In the following graphs, we plot the prices of MBS, IO, and PO as a function of the instantaneous short rate. Since there is no closed-form solution to price these contracts, the durations and convexities are only numerically obtained.

To ensure the value biases in Figure 1 are not artificially created by unreasonable prepayment assumption we examine the distribution of the prepayment speed generated during the 500-path Monte Carlo simulations. Figure 6 below plots the quartiles and the 5 percentile tail prepayment rates. The range is consistent with the real-life MBS prepayment experiences during different interest rate experiences. Of course, the sensitivity of the prepayment rate with respect to market interest rates does have an impact on the values. However, by applying different values for the sensitivity parameter (h), we find the above pattern of the results remains the same.

Figure 6. Mortgage Survival Distribution

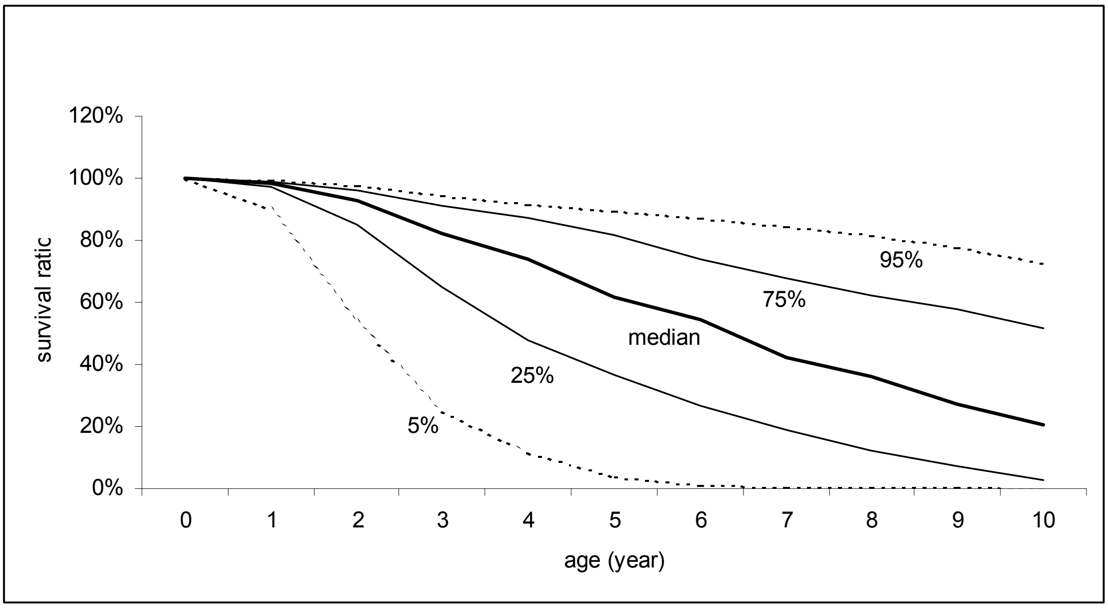
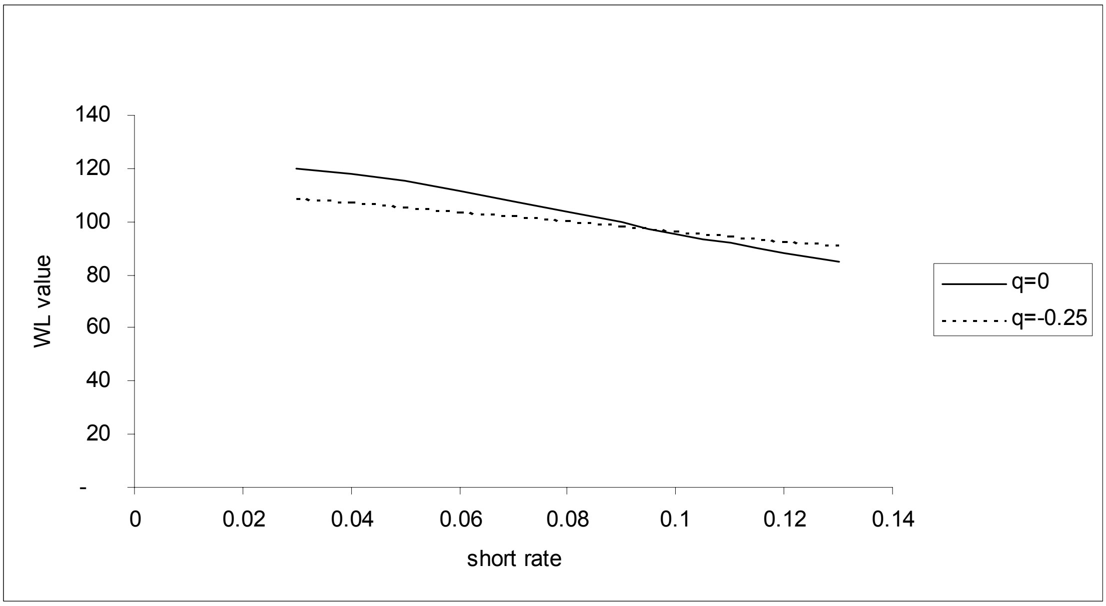


Figure 7. Interest Rate Sensitivity: MBS



In Figure 7, we observe that the MBS has little convexity, under both real and risk-neutral measures. Although the pricing level is similar, the duration of the risk-neutral model is almost double that of the equilibrium model. Hedging duration based on the risk-neutral model can actually reverse the duration. It not only increases hedging costs, but it also can expose the investor to interest rate risk in the *opposite* direction.

Figure 8 plots the IO interest rate sensitivity under the two measures. We observe that not only is the slope flatter, but the curvature is also less. The negative convexity of the risk-neutral process is much stronger than that of the equilibrium model. This can cause mortgage servicers to over-hedge the convexity. That is, instead of adjusting the convexity to zero, one can over-hedge to the extent that the convexity is positive. Should the investor choose not to hedge, the consequence is a much higher capital reserve based on the lower value suggested by the risk-neutral model at both high and low interest rate shocks.

Figure 8. Interest Rate Sensitivity: Interest Only

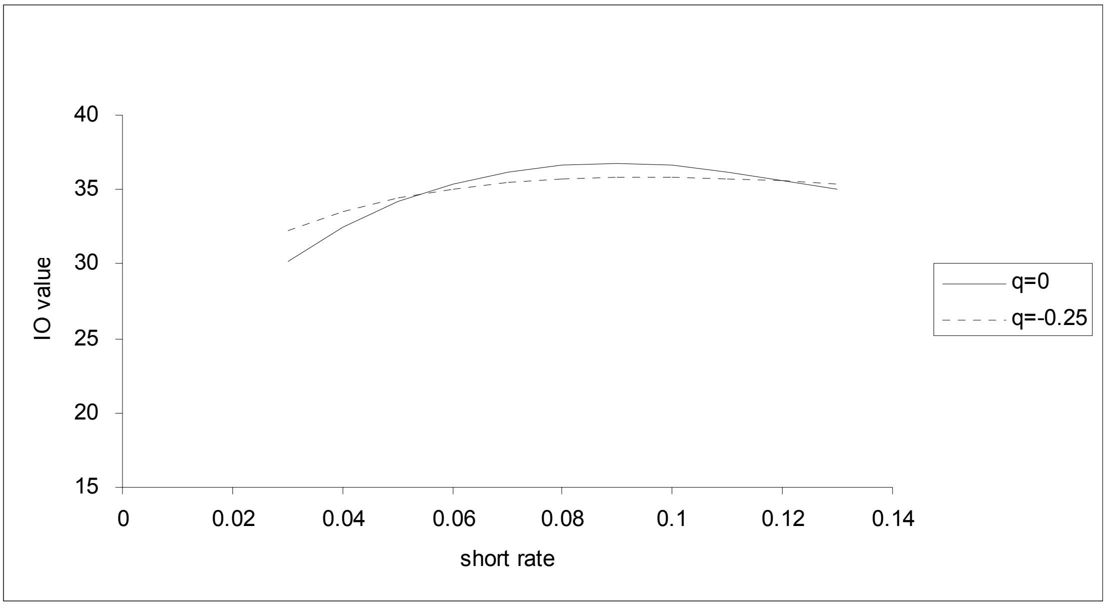
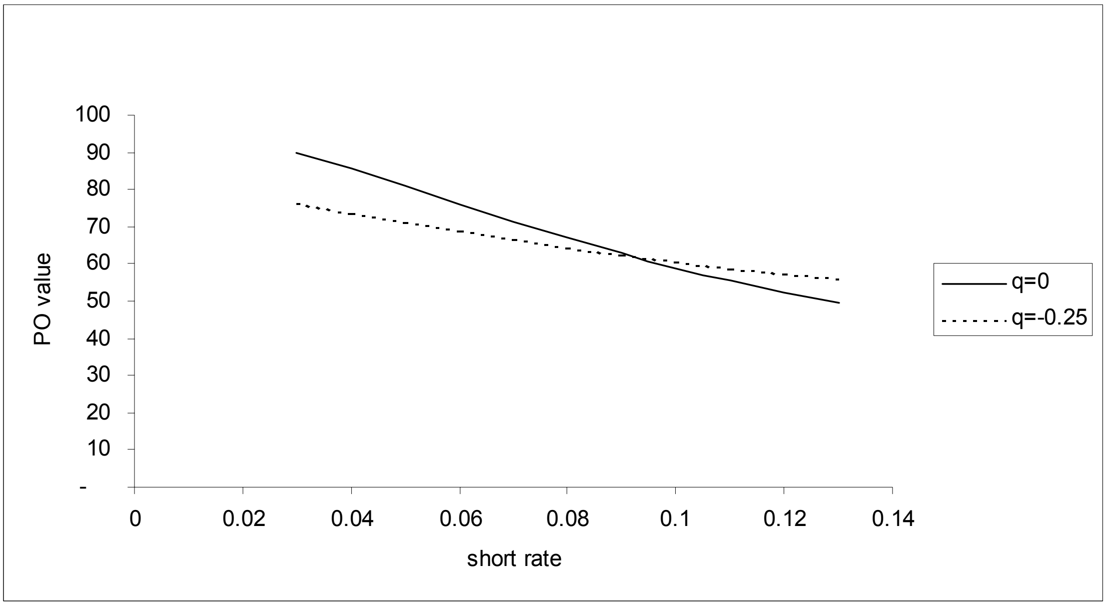


Figure 9 plots the PO interest rate sensitivity under the two measures. The results are very similar to those of the MBS. The risk-neutral pricing result has a longer duration than that of the (real) equilibrium model and both have little convexity.

Figure9. Interest Rate Sensitivity: Principal Only



Based on these interest rate sensitivities, the duration estimated by the risk-neutral assumption shows considerable difference from the equilibrium model. From the results below, we found the bias is most significant in IO strips, where the bias could be up to 22% to 75% in the most relevant risk-aversion range. Such sizable difference can cause material impact to the investor's hedging cost as well as the actual interest rate risk exposure.

Table 2 Duration of Securities

	Duration		
q	MBS	IO	PO
0	2.11	-0.74	3.56
-0.1	1.96	-0.76	3.36
-0.25	1.79	-0.90	3.23
-0.5	1.82	-1.29	3.58
-1	2.17	-2.65	4.74
	% Bias in price when assuming $q = 0$		
-0.1	-7%	2%	-6%
-0.25	-15%	22%	-9%
-0.5	-14%	75%	1%
-1	3%	258%	33%

Table 3 shows the similar calculations for the convexity. These results reveal even more severe differences when investors' risk aversion behavior is incorporated. Different from the duration case, the most severe convexity difference appears in the PO strip piece, which understate the convexity by over 100 percent. For IO strips like the mortgage servicing rights, the convexity has been understated by about 30 percent.

Table 3 Convexity of Securities

	Convexity		
q	MBS	IO	PO
0	-0.07	-0.39	0.10
-0.1	-0.02	-0.45	0.20
-0.25	-0.05	-0.53	0.20
-0.5	-0.05	-0.58	0.25
-1	0.33	-0.46	0.75
	% Bias in price when assuming $q = 0$		
-0.1	-71%	14%	107%
-0.25	-27%	34%	110%
-0.5	-27%	47%	156%
-1	-589%	18%	672%

6. SUMMARY

The typical approach involves simulating interest rates under risk-neutral space but attaches with actuarially estimate prepayment rate functions poses an internal model inconsistency. This paper is the first to directly evaluate the robustness of such a common practice. Following Cox, Ingersoll, Ross (1985) general equilibrium setting, we use a market risk price measure to back out the real interest rate process from the observed yield curve. This allows us to model the mortgages using the consistent real measures of interest rate and the prepayment rate as a function of the underlying interest rate process.

Numerical examples are provided to investigate the size of the biases introduced by the typical models' inconsistent treatment of interest rate process and prepayment equation under different measurement spaces. The results show that while the mixed approach may provide reasonable estimate for a par mortgage, it tends to generate pricing bias for premium and discount MBSs. The more severe biases are found to be on the risk management area. The duration measure can be biased by up to 20 percent. The bias can be on different direction depending on the particular interest hold in portfolio. The convexity bias is even greater. These biases in risk estimates can introduce failure in hedging against interest rate and prepayment risks. Meanwhile, over hedging also means extra hedging costs.

The applications of this model in MBS pricing is fruitful. First, the use of the real measure enables easily extending the model to incorporate house price volatility into the existing interest rate risk framework. Secondly, the use of the real measure can correctly model prepayment behavior. Lastly, the use of the real measure allows us to perform accurate Value-at-Risk estimates, which rely on the tails of the distributions.

A natural extensions of this paper is the incorporation of the house price process and default risk for the impact of mortgage guarantees. In the risk-neutral space, the underlying collateral property value would follow the identical process. This prevents us from capturing the geographical deviation of real estate market performance. In reality, the real estate market is very local and tends to follow a long business cycle. At any point in time, we can find local markets at different stages of a cycle. Adopting the risk-neutral property value process fails to reflect this important real estate market reality.

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