# Cross Hedging Under Multiplicative Basis Risk*


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**Abstract**

In incomplete financial markets, managing price risk has to rely on cross hedging. This creates basis risk. Rather than modeling basis risk through an additive noise term, this paper proposes an alternative view where a *multiplicative* noise term captures basis risk. Consequently, basis risk is proportional to the price level. The multiplicative modeling has various advantages over its additive counterpart. The decision maker's prudence is essential: If the spot price is the product of the forward price and basis risk, positive prudence is a necessary and sufficient condition for underhedging in an unbiased market. If the forward price is the product of the spot price and basis risk, non-negative prudence is a sufficient condition for underhedging. Both results are robust to the introduction of quantity risk.



JEL classification: D81; G11

Keywords: risk management, cross hedging, basis risk, quantity risk, prudence



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# Cross Hedging Under Multiplicative Basis Risk

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# 1 Introduction

In an incomplete financial market, a risk-averse decision maker does not have the opportunity to trade hedging instruments that perfectly replicate the initial exposure to some price risk. Hence, the decision maker has to manage price risk with a cross hedge which creates basis risk. Consider three examples: If the composition of an equity portfolio differs from the composition of the market index and the investor hedges with index futures, there will be basis risk. The same applies to a farmer growing a particular grade of corn if the hedging instrument relates to another grade of corn. Investment banks also bear basis risks when they sell customized derivatives products to corporate clients and cross hedge with standardized exchange-traded financial derivatives.

This paper focuses on how the optimal forward cross hedge is affected by the nature of basis risk. Most cross hedging models assume that the spot price is equal to the forward price plus a noise term capturing basis risk. Since Benninga et al. (1983), this additive type of dependence has been used extensively in theoretical cross hedging studies.[1] For any risk-averse decision maker, the optimal cross hedge in an unbiased forward market is a full hedge. Briys et al. (1993) derive the optimality of an underhedge assuming the reverse relationship where the forward price is a linear function of the spot price and basis risk.

Under additive basis risk, the amount of basis risk is independent of the level of the forward price or the spot price. Apart from the analytical convenience and the striking elegance of the results achieved, it is difficult to justify this assumption. In particular, the assumption of additive basis risk is unsatisfactory because the economic conflict between managing price risk and, in doing so, generating basis risk is defined away in the first place. Hence, the economic conflict remains hidden so that it is not surprising that the optimal decision ignores basis risk. Therefore, this paper proposes an alternative view by focusing on a *multiplicative* relationship between basis risk and the forward or spot

[1] See Benninga et al. (1984), Broll et al. (1995), Chang and Wong (2003), among others.

price. It follows that the amount of basis risk is no longer independent of the price level but is proportional to it. This paper analyzes two types of multiplicative dependence: In the first type, the spot price is the product of the forward price and a conditionally independent basis risk. This specification is consistent with the log-log model which is widely used in the empirical literature on futures pricing. The second type is based on the reverse assumption: The forward price is the product of the spot price and an independent basis risk. This reverse assumption is consistent with the standard approach to futures pricing. Both assumptions seem particularly attractive when correlations between prices behave asymmetrically in that correlations are higher in a market downturn than in an upturn. (Ang and Chen (2002) provide evidence for such a correlation asymmetry in the US stock market.) The additive modeling of basis risk is neither consistent with the log-log model nor the standard approach to futures valuation nor can it capture asymmetries in correlations. Hence, modeling basis risk in a multiplicative way seems more attractive.

For an unbiased forward market, the results are the following: Given the first type of dependence, the optimal hedge ratio is determined by the decision maker's absolute prudence as define by Kimball (1990): Positive absolute prudence is a necessary and sufficient condition for underhedging; a full hedge is optimal under quadratic utility only. For the second type of dependence, non-negative absolute prudence is a sufficient condition for underhedging. Both results remain intact if the decision maker's exposure to price risk is itself random (quantity risk).[2] These results demonstrate that optimal cross hedging under multiplicative basis risk does not simply mirror optimal decisions under additive risk.[3]

Mahul (2002) proposes the first type of multiplicative basis risk and focuses on the hedging role of futures and options in the first-best hedging portfolio. This paper takes a different route by explicitly deriving the optimal forward position under multiplicative

[2]Moschini and Lapan (1995) analyze optimal hedging of jointly normally distributed price risk, basis risk and quantity risk under constant absolute risk aversion.

[3]Focusing on background risk, Franke et al. (2006) show that multiplicatively combined risks lead to different risk taking if compared to additively combined risks.

basis risk and by also considering the reverse type of multiplicative dependence.[4]

Brown and Toft (2002) consider the optimal hedging strategy of a value-maximizing firm facing price and quantity risk. While their analysis incorporates options and exotic hedges[5], this comes at the cost of assuming jointly (log-) normally distributed prices and quantities and the absence of basis risk. In contrast, the present paper is more general in that all results can be derived without imposing any specific distributions.

The paper is organized as follows: Section 2 delineates the framework of analysis. Section 3 presents the two types of multiplicative dependence in detail and contrasts them to the additive types. Optimal forward hedging under multiplicative basis risk is analyzed in Section 4. Section 5 characterizes optimal forward hedging if there is quantity risk in addition to price risk and basis risk. Section 6 concludes. All proofs are in the Appendix.

## 2 The basic model

The analysis is based on the standard two-date expected utility hedging model: At date 0, the risk-averse decision maker has a given exposure $Q$ to a price risk $\tilde{P}$.[6] At date 1, quantity $Q$ is sold at the random future spot price $\tilde{P}$. At date 0, the decision maker can sell an amount $X$ in an infinitely divisible forward contract at the given forward price $f$ and will unwind the forward position at date 1 at the random forward price $\tilde{F}$.[7] Final wealth is thus given by $\tilde{W} = \tilde{P}Q + (f - \tilde{F})\,X$. Preferences are summarized by a utility function $U(W)$ that is at least three times continuously differentiable and exhibits risk aversion, $U'(W) > 0$, $U''(W) < 0$.[8] In addition, $\lim_{W \to 0} U'(W) \to +\infty$ and $\lim_{W \to +\infty} U'(W) \to 0$.

[4] Considering non-linear instruments such as options is outside the scope of this paper. Hentschel and Kothari (2001) show that linear contracts such as forwards and swaps seem to play a dominant role at least in corporate hedging.

[5] The earliest paper to consider exotic hedges is Ware and Winter (1988).

[6] Henceforth, random variables have a tilde ($\tilde{}$) while their realizations do not. A star ($^*$) indicates an optimized level.

[7] Of course, $\tilde{P}$ and $\tilde{F}$ are assumed to have positive realizations only, $P > 0$, $F > 0$.

[8] It seems reasonable to assume that privately held, owner-managed firms behave in a risk-averse manner. But even firms with well-diversified shareholders and separation of ownership and control tend to behave as if they were risk-averse. This can be attributed to agency considerations (Stulz, 1984; DeMarzo and Duffie,

At date 0, the decision maker's problem is to maximize expected utility, $\mathrm{E}[U(\tilde{W})]$, by choosing a hedging position, $X$, in order to manage price risk. Since the utility function is concave and $U'(W) \in (0, \infty)$, the first-order condition for the optimal forward position, $X^*$, is necessary and sufficient for a unique and interior optimum. It is given by

$$\mathrm{E}\Big[U'(\tilde{W}^*)\left(f - \tilde{F}\right)\Big] \;=\; 0\,. \tag{1}$$

In order to focus on the hedging role of forward contracts, the forward market is assumed to be unbiased, $f = \mathrm{E}[\tilde{F}]$.[9] Consequently, the first-order condition reduces to $-\operatorname{cov}(U'(\tilde{W}^*), \tilde{F}) = 0$. In the case of backwardation or contango, $f \neq \mathrm{E}[\tilde{F}]$, there will be speculative positions which could be easily incorporated into the model.

It is assumed that the decision maker cannot eliminate price risk completely from final wealth since the capital market is imperfect in the sense that $\tilde{P}$ and $\tilde{F}$ are not perfectly correlated. There may be various reasons for this incompleteness. One reason is that there is no forward contract available for the underlying price risk because of market failure for fear of moral hazard, insufficient size of the potential market or government regulations such as exchange restrictions. In this case, the decision maker will have to rely on a forward contract on another underlying. A second reason could be the existence of a timing mismatch: The forward position will have to be liquidated at date 1 but expires at some later date.[10] A third reason is the existence of default risk in forward contracts: If the counterparty defaults, the forward contract is ineffective at least for $f > F$ (for the short hedger), leaving the hedger exposed to price risk.[11]

1995; Morellec and Smith, 2005), corporate taxes and costs of financial distress (Smith and Stulz, 1985; Graham and Smith, 1999; Graham and Rogers, 2002) as well as other capital market imperfections (Stulz, 1990; Froot et al., 1993). Brown and Toft (2002) consider a value-maximizing firm facing deadweight costs that are convex in the firm's profits, thereby concavifying the firm's objective function. Against this background, the results derived here are also valid for corporate risk management applications.

[9] Benninga and Oosterhof (2004) show that the representative agent does not necessarily have to be risk neutral in order to ensure that the forward market is unbiased.

[10] Liquid forward markets are characterized by a small number of maturity dates only. If the decision maker's exposure does not fit into the maturity structure available in the market, there will be a timing mismatch. The same applies even more pronounced to futures contracts.

[11] See Doherty and Schlesinger (1990) for an insurance purchasing problem where the insurer might default.

The basis is defined as the difference between the forward price $\tilde{F}$ and the future spot price $\tilde{P}$. Since the forward contract does not perfectly match the decision maker's price risk, the basis is stochastic, hence there is basis risk in addition to price risk. Consequently, risk management has to rely on *cross hedging* or *indirect hedging*. Whatever the forward position, final wealth is never deterministic if there is basis risk. In order to formalize basis risk, an additional random variable, denoted $\tilde{g}$ or $\tilde{\gamma}$, has to be incorporated into the model. This can be done either by assuming that the spot price is a function of the forward price and basis risk, $\tilde{P} = p(\tilde{F}, \tilde{\gamma})$, or by assuming that the forward price is a function of the spot price and basis risk, $\tilde{F} = \phi(\tilde{P}, \tilde{g})$.[12]

## 3 Additive versus multiplicative basis risk

This section briefly reviews additive basis risk and presents two types of multiplicative basis risk as an alternative.

So far, the literature focused on *additive* relationships between basis risk and either $\tilde{F}$ or $\tilde{P}$. Building on Benninga et al. (1983), Lence (1995) considers $\tilde{P} = p(\tilde{F}, \tilde{\gamma}) = \kappa\,\tilde{F} + \omega(\tilde{\gamma})$ for an arbitrary constant $\kappa$ and some arbitrary function $\omega(\cdot)$ with $\mathrm{E}[\omega(\tilde{\gamma})] = 0$. He shows that conditional independence of $\tilde{F}$ from $\tilde{\gamma}$ is necessary and sufficient for the optimality of a full hedge in an unbiased forward market, $X^* = \kappa\, Q$.[13] This full hedge is optimal for any risk-averse decision maker. The reverse relationship is given by $\tilde{F} = \phi(\tilde{P}, \tilde{g}) = k\,\tilde{P} + w(\tilde{g})$ where $\tilde{P}$ and $\tilde{g}$ are stochastically independent and $w(\cdot)$ is a linear function.[14] Briys et al. (1993), Broll et al. (1995) and others show that an underhedging position is optimal for all risk averters, $X^* < Q/k$ for $k > 0$. Under the additive relationships, basis risk is independent of the level of $\tilde{F}$ and $\tilde{P}$.

[12] Both cases have to be considered separately since $p(\cdot)$ and $\phi(\cdot)$ are not necessarily invertible.

[13] $\tilde{x}$ is said to be conditionally independent of $\tilde{y}$ if $\mathrm{E}[\tilde{x}|y] = \mathrm{E}[\tilde{x}]\,\forall\, y$. Under mild regularity conditions, this is equivalent to $\mathrm{cov}(\tilde{x}, h(\tilde{y})) = 0$ for all functions $h(\cdot)$. See Ingersoll (1987, p. 15).

[14] This is not exactly the reverse relationship because independence is stronger than conditional independence. In addition, the linearity assumption imposed on $w(\cdot)$ is not needed for $\omega(\cdot)$.

As an alternative, this paper proposes two multiplicative relationships, labeled M.1 and M.2:

**Assumption M.1:** $\tilde{P} \;=\; \alpha + \beta \tilde{F}(1+\tilde{\gamma}),$

where $\tilde{\gamma}$ is conditionally independent of $\tilde{F}$. $\tilde{\gamma}$ has support in the interval $[\underline{\gamma}, \overline{\gamma}]$ with $-1 < \underline{\gamma} < 0 < \overline{\gamma} < \infty$ and $\mathrm{E}[\tilde{\gamma}] = 0$. The support of $\tilde{F}$ is a subset of $[\underline{F}, \overline{F}]$ with $0 < \underline{F} < \overline{F} < \infty$. Finally, $\alpha > -\beta \underline{F}(1+\underline{\gamma})$ and, without loss of generality, $\beta > 0$. These assumptions imply $P > 0$ and $\mathrm{E}[\tilde{P}] = \alpha + \beta \mathrm{E}[\tilde{F}]$. Under M.1, the spot price $\tilde{P}$ is a multiplicative combination of basis risk $\tilde{\gamma}$ and the forward price $\tilde{F}$. Basis risk from $\tilde{\gamma}$ is proportional to $F$.

**Assumption M.2:** $\tilde{F} \;=\; a + b \tilde{P}(1+\tilde{g}),$

where $\tilde{g}$ and $\tilde{P}$ are stochastically independent. The support of $\tilde{g}$ is a subset of $[\underline{g}, \overline{g}]$ with $-1 < \underline{g} < 0 < \overline{g} < \infty$ and $\mathrm{E}[\tilde{g}] = 0$. The support of $\tilde{P}$ is contained in $[\underline{P}, \overline{P}]$ with $0 < \underline{P} < \overline{P} < \infty$. In addition, $a > -b \underline{P}(1+\underline{g})$ and, again for simplicity, $b > 0$.[15] $F > 0$ and $\mathrm{E}[\tilde{F}] = a + b \mathrm{E}[\tilde{P}]$ follow. Under M.2, the forward price $\tilde{F}$ is a multiplicative combination of basis risk $\tilde{g}$ and the spot price $\tilde{P}$. Basis risk, arising from $\tilde{g}$, is proportional to $P$.

There are two important technical remarks to be made here: Firstly, M.1 cannot be derived from M.2 and vice versa. $\tilde{\gamma}$ captures basis risk under M.1 and is conditionally independent of $\tilde{F}$ such that it cannot be independent of $\tilde{P}$ at the same time as required under M.2. The same applies to M.2 where $\tilde{g}$ is independent of $\tilde{P}$ so that it cannot be conditionally independent of $\tilde{F}$ as M.1 requires. Secondly, the model is very flexible since the (marginal) distributions of $\tilde{F}$ and $\tilde{P}$ do not have to be restricted for as long as their difference satisfies M.1 or M.2. The same applies to the distributions of $\tilde{g}$ and $\tilde{\gamma}$.

Multiplicative basis risk is more attractive than additive basis risk in at least two respects. Firstly, M.2 is consistent with the standard approach to forward valuation.[16]

[15] All results can also be derived for $\beta < 0$ and $b < 0$, with some obvious modifications.

[16] See Working (1949).

This approach employs a simple no arbitrage argument to suggest that, at date 1, the forward price is given by $F = P\, e^{(r-\theta)}$ for a forward contract expiring at some later date 2. In this relationship, $r$ denotes the risk-free interest rate between dates 1 and 2, $\theta$ denotes either the convenience yield (for a commodity contract) or the dividend yield (for a financial contract) between these dates. Now, assume that this standard valuation approach is valid at date 1. From the perspective of date 0, neither $P$ nor $r$ nor $\theta$ are known with certainty. Hence, they have to be treated as random variables such that the valuation relation becomes $\tilde{F} = \tilde{P}\, e^{(\tilde{r}-\tilde{\theta})}$. Setting $a = 0$, $b = \mathrm{E}[e^{(\tilde{r}-\tilde{\theta})}]$ and $(1+\tilde{g}) = e^{(\tilde{r}-\tilde{\theta})}/\mathrm{E}[e^{(\tilde{r}-\tilde{\theta})}]$ demonstrates that the standard approach to forward pricing is fully compatible with M.2.[17] This is a clear advantage of M.2 over the corresponding additive relationship, $\tilde{F} = k\,\tilde{P} + w(\tilde{g})$, which is obviously not consistent with the standard valuation approach.

Secondly, M.1 is closely related to the log-log model which is widely used in empirical studies on the relationship between the spot price $\tilde{P}$ and the forward price $\tilde{F}$.[18] In the log-log model, it is assumed that $\ln(\tilde{P}) = c + \ln(\tilde{F}) + \tilde{\epsilon}$ where $\tilde{F}$ and $\tilde{\epsilon}$ are stochastically independent and $\mathrm{E}[\tilde{\epsilon}] = 0$. Setting $\alpha = 0$, redefining $\ln(\beta) + \mathrm{E}[\ln(1+\tilde{\gamma})] = c$ and $\ln(1+\tilde{\gamma}) - \mathrm{E}[\ln(1+\tilde{\gamma})] = \tilde{\epsilon}$ and taking the natural logarithm of M.1 yields $\ln(\tilde{P}) = \ln(\beta) + \ln(\tilde{F}) + \ln(1+\tilde{\gamma}) = c + \ln(\tilde{F}) + \tilde{\epsilon}$. It follows that M.1 is in line with the log-log model whereas the respective additive relationship, $\tilde{P} = \kappa\,\tilde{F} + \omega(\tilde{\gamma})$, is not.

Depending on the particular hedging application, there might be a rationale to prefer one particular relationship over the others.[19] For example, suppose that correlations between prices behave asymmetrically, depending on the direction of the price change. As Ang and Chen (2002) show for the US stock market, correlations between individual stocks

[17] Notice that the joint distribution of $\tilde{r}$ and $\tilde{\theta}$ does not have to be restricted since $e^{(r-\theta)}$ is always positive so that $(1+g) > 0$ as is required by M.2.

[18] See the surveys by Chow et al. (2000) and Lien and Tse (2002).

[19] No econometric tests have so far been performed that compare the suitability of M.1, M.2 or one of the additive specifications for a particular cross hedging problem. While such a test is beyond the scope of this paper, it is clear that both M.1 and M.2 are non-nested with each of the additive specifications; furthermore, M.1 and M.2 are non-nested unless $a = \alpha = 0$. Gourieroux and Monfort (1994) and Pesaran and Weeks (2001) provide surveys on model selection and hypotheses testing with non-nested models; see also McAleer (1995).

and the aggregate market have been larger in market downturns than in market upturns.[20] If an investor uses forward contracts on a market index to hedge an individual equity portfolio, this has the following implications: If equity prices decline, the (conditional) correlation between the portfolio value and the forward price is larger and, therefore, basis risk is smaller. But if (conditional) correlation decreases when equity prices rise, basis risk grows as well. This is perfectly in line with assumptions M.1 and M.2 as basis risk is proportional to the price level. In contrast, additive basis risk cannot capture the effects of such correlation asymmetries.

## 4 Optimal cross hedging under multiplicative basis risk

This section presents the optimal forward positions under assumptions M.1 and M.2. The main results are summarized in two propositions.

Under M.1, price risk $\tilde{P}$ can be interpreted as a bundle of tradable forward price risk $\tilde{F}$ and untradable basis risk $\tilde{\gamma}$. Final wealth under M.1 is given by

$$\tilde{W} \;=\; \tilde{\gamma}\tilde{F}\beta Q + \xi(\tilde{F}, X)\,, \qquad (2)$$

where $\xi(\tilde{F}, X) = \tilde{F}(\beta Q - X) + (\alpha Q + fX)$ does not depend on $\tilde{\gamma}$. The first summand in (2) shows that the decision maker's exposure to untradable basis risk $\tilde{\gamma}$ is independent of the forward position $X$. However, the multiplicative relationship between $\tilde{\gamma}$ and $\tilde{F}$ establishes an indirect link through $\xi(\tilde{F}, X)$.[21] Therefore, basis risk $\tilde{\gamma}$ will generally affect the optimal forward position even though it is not directly related to it.

[20] Bekaert and Wu (2000) point to a related phenomenon in the Japanese equity market. See also Kroner and Ng (1998). Patton (2006) shows that the correlation between the JPY/USD and the DEM/USD exchange rates has been higher when the USD appreciated than when the USD depreciated against these two currencies.

[21] This multiplicative combination of $\tilde{\gamma}$ and $\tilde{F}$ is the reason why basis risk is not identical to an additive background risk. For the case of an additive independent background risk, Briys et al. (1993) show that full hedging in an unbiased forward market is optimal for any risk-averse decision maker.

Before presenting the main result of the paper in Proposition 1, it is worth taking a closer look at the result derived by Lence (1995). Holthausen (1979) shows that full hedging is optimal if there is no basis risk and the forward market is unbiased.[22] If basis risk is additive as in Lence's (1995) model, full hedging is still optimal for all risk-averse decision makers. Therefore, in his framework, the size of the basis risk or whether it exists at all has no impact on the optimal decision. The reason for this somewhat counterintuitive result is that, by assumption, price risk $\tilde{P}$ is decomposed into a tradable part $\tilde{F}$ and an independent, non-tradable part that captures basis risk, $\tilde{P} = \kappa \tilde{F} + \omega(\tilde{\gamma})$. Since there is neither a direct nor an indirect connection between basis risk and the forward position, the decision maker can only use the forward market to entirely sell the tradable part of the $\tilde{P}$-risk which is the risk from $\tilde{F}$. The particular assumption on the additive nature of basis risk is unsatisfactory because the economic conflict between managing price risk arising from $\tilde{P}$ and, in doing so, generating additional risk in the form of basis risk is defined away in the first place. Hence, the economic conflict is not accounted for under this type of additive basis risk. It is therefore not surprising that the optimal decision ignores basis risk.

In sharp contrast, Proposition 1 clearly shows that there is such a conflict under M.1. More importantly, it also shows how the decision maker's preferences determine the optimal decision against the background of this conflict: The optimal forward position depends on the decision maker's (absolute) prudence, defined by Kimball (1990) as $-U'''(\cdot)/U''(\cdot)$.

**Proposition 1** *Suppose that the forward market is unbiased and that M.1 holds. The optimal forward position is a short hedge, $X^* > 0$. An underhedging position, $X^* < \beta Q$, is optimal if and only if the decision maker is prudent, $U'''(\cdot) > 0$.*[23]

[22] Benninga and Oosterhof (2004) show that this holds even if the decision maker's individual valuation of the forward contract differs from the market's valuation.

[23] As shown in the Appendix, $X^* > [=] \beta Q$ if and only if $U'''(\cdot) < [=] 0$.

All risk-averse decision makers optimally sell at least part of their exposure to $\tilde{P}$ in the forward market such that $X^* > 0$. More importantly, a prudent decision maker optimally chooses an underhedging position. The forward position that minimizes the variance of $\tilde{W}$ is the full hedge, $X^{vm,1} = \beta Q$.[24] A full hedge is optimal only for quadratic utility.

In order to see the intuition behind Proposition 1, rewrite the first-order condition for $X^*$ as $\mathrm{E}\{\mathrm{E}[U'(\tilde{\gamma}F\beta Q + \xi(F, X^*))|F](f - \tilde{F})\} = 0$, applying the law of iterated expectations. Using a Taylor expansion of $U'(\cdot)$ around $\xi(F, X)$, expected marginal utility for a given $F$ and a small $\tilde{\gamma}$-risk can be written as

$$\mathrm{E}\Big[U'\Big(\tilde{\gamma}F\beta Q + \xi(F,X)\Big)\Big|F\Big] \quad = \quad U'\Big(\xi(F,X)\Big) + \mathrm{var}(\tilde{\gamma})\,\frac{(F\beta Q)^2}{2}\,U'''\Big(\xi(F,X)\Big)\,. \qquad (3)$$

Since marginal utility is decisive for hedging, the second term on the RHS of (3) indicates that the impact of basis risk $\tilde{\gamma}$ on the optimal forward position $X^*$ depends on $F$ and on $U'''(\cdot)$. This dependence of $X^*$ on $F$ establishes the indirect link between basis risk and the optimal forward position. If and only if utility is quadratic, basis risk is ignored such that full hedging is optimal. Whenever marginal utility is not linear, basis risk is taken into account such that it affects the optimal forward position.

Proposition 1 shows how non-linear marginal utility determines the optimal decision: Positive prudence, $U'''(\cdot) > 0$, is a necessary and sufficient condition for the optimality of an underhedging position. Positive prudence is a commonly accepted property of utility functions and has become an integral part of the literature on behavior under uncertainty as based on the expected utility paradigm.[25] Basically, positive prudence implies that the decision maker has a precautionary incentive to avoid particularly low realizations of final wealth. It also implies that the motivation to increase final wealth in such states is stronger than under quadratic utility.[26] In order to see how prudence affects the optimal

[24] Conditional independence of $\tilde{\gamma}$ from $\tilde{F}$ together with $\mathrm{E}[\tilde{\gamma}] = 0$ implies $\mathrm{cov}(\tilde{F}, \tilde{F}\tilde{\gamma}) = 0$ such that $\mathrm{var}(\tilde{W}) = (\beta Q - X)^2\,\mathrm{var}(\tilde{F}) + (\beta Q)^2\ \mathrm{var}(\tilde{\gamma}\tilde{F})$. $X^{vm,1} = \beta Q$ directly follows.

[25] Positive absolute prudence is a necessary condition for decreasing absolute risk aversion. Further arguments in favor of positive prudence can be found in Gollier (2001, Ch. 16), among others.

[26] Brown and Toft (2002) argue that a value-maximizing firm hedges because it might face costly states

decision, it is useful to start at the full hedge position, $X^{vm,1} = \beta Q$. At $X^{vm,1}$, states with very low $W$ are characterized by a strongly negative realization of $\tilde{\gamma}$ together with a highly positive realization of $\tilde{F}$ as follows from $\tilde{W}(X = X^{vm,1}) = \tilde{\gamma}\,\tilde{F}\,\beta Q + \alpha Q + f\,X^{vm,1} = (\tilde{\gamma}\,\tilde{F} + f)\,X^{vm,1} + \alpha Q$. As (2) indicates, generating additional final wealth in states with high $F$ requires selling less forward contracts since this is the only possibility to increase $\partial W/\partial F$. Thus, the optimal forward position of a prudent decision maker is equivalent to an underhedge, $X^* < \beta Q = X^{vm,1}$.

Kimball (1990) shows that prudence plays a crucial role in the presence of an additive background risk. Proposition 1 shows that the decision maker's prudence is also essential in the presence of a multiplicative basis risk. Benninga et al. (1985) show that prudence is also important if there is tradable price risk and independent, untradable quantity risk, but no basis risk.[27] Lence (1995) mentions briefly that the optimal hedge ratio in the log-log model depends on the utility function but he does not specify the dependence. Proposition 1 does not only confirm his statement but also shows exactly *how* the optimal forward position depends on the decision maker's preferences.

Now, consider M.2. Under M.2, tradable forward price risk $\tilde{F}$ is like a package of price risk $\tilde{P}$ and basis risk $\tilde{g}$. While the exposure to $\tilde{P}$ is exogenously given by $Q$, the exposure to the package that forms $\tilde{F}$ is endogenously determined via the forward position $X$. In other words, if the decision maker seeks protection against fluctuations in $\tilde{P}$ by trading $\tilde{F}$, this creates exposure to basis risk $\tilde{g}$. Under M.2, final wealth can be written as

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of nature where profits are very low. In a similar spirit, a prudent decision maker focuses on states with very low final wealth.

[27] In the absence of basis risk, $\tilde{P} = \alpha + \beta\tilde{F}$. To connect the model analyzed here to the Benninga et al. (1985) model, define a random variable $\tilde{z}$, independent of $\tilde{P}$ with $\mathrm{E}[\tilde{z}] = 1$ and $z \geq 0$. Then, the exposure is given by $Q\tilde{z}$ and final wealth becomes $\tilde{W} = (\tilde{z} - 1)(\alpha + \beta\tilde{F})Q + \xi(\tilde{F}, X)$ where $\xi(\tilde{F}, X)$ is defined as in (2). The first summand here, $(\tilde{z} - 1)(\alpha + \beta\tilde{F})Q$, captures quantity risk whereas the first summand in (2) captures basis risk. Apart from this, the problems are isomorphic. This explains why Proposition 1 is in line with the result derived by Benninga et al. (1985). Dependent price and quantity risk is analyzed by Brown and Toft (2002) for a value-maximizing firm and by Wong (2003) in an expected utility framework, but neither model incorporates basis risk.

$$\tilde{W} \; = \; \tilde{P}\Big(Q - bX\Big) - \tilde{g}\tilde{P}\Big(bX\Big) + X\left(f - a\right). \tag{4}$$

The second term on the RHS of (4) shows that basis risk $\tilde{g}$ enters final wealth only in multiplicative combination with $\tilde{P}$ and $X$. There is no basis risk if there is no forward position, $X = 0$. At full hedging, $X = Q/b$, the isolated effect of price risk $\tilde{P}$ on $\tilde{W}$ as represented by the first term on the RHS of (4) is eliminated. However, any forward position exposes final wealth to basis risk $\tilde{g}$. Hence, there is a conflict between reducing price risk $\tilde{P}$ and avoiding basis risk $\tilde{g}$: The first term on the RHS of (4) favors a full hedging position, the second a forward position of zero. The next result characterizes the optimal forward position:

**Proposition 2** *Suppose that the forward market is unbiased and that M.2 holds.*

*a) The optimal forward position is a short hedge,* $X^* > 0$.

*b) If* $U'''(\cdot) \geq 0$*, the optimal forward position is an underhedge,* $X^* < Q/b$.

Part a) of Proposition 2 shows that all risk-averse decision makers will optimally hedge at least part of the price risk $\tilde{P}$ by choosing a positive forward position, $X^* > 0$. This causes final wealth to depend on basis risk $\tilde{g}$ as well. The decision maker enjoys a gain from diversification: $X^* > 0$ is the same as selling part of one risk ($\tilde{P}$) and acquiring another risk ($\tilde{g}\tilde{P}$) which is uncorrelated with the first.

Part b) of Proposition 2 states that non-negative absolute prudence is a sufficient condition for underhedging under M.2. To see why, consider a prudent decision maker who starts at full hedging. Final wealth, expressed in terms of tradable forward price risk $\tilde{F}$ and untradable basis risk $\tilde{g}$, is then given by

$$\tilde{W}\Big(X = Q/b\Big) \; = \; \frac{Q}{b}\left[\frac{-\tilde{g}(\tilde{F} - a)}{1 + \tilde{g}} + f - a\right]. \tag{5}$$

Since $b$, $Q$, $(F - a) > 0$ in all states, final wealth is low when the realizations of $\tilde{g}$ and $\tilde{F}$ are both high. Positive prudence therefore creates a strong motive to generate additional wealth in these unfavorable states which requires a reduction of the forward position. Hence, underhedging is optimal.

The variance-minimizing position under M.2 is given by[28]

$$X^{vm,2} \;=\; \frac{Q}{b}\left(\frac{\operatorname{var}(\tilde{P})}{\mathrm{E}[\tilde{P}^2]\operatorname{var}(\tilde{g}) + \operatorname{var}(\tilde{P})}\right) \;=\; \frac{Q}{b}\,K \tag{6}$$

where $K$ is a positive constant below unity. Hence, the variance-minimizing forward position is always an underhedging position as implicitly claimed in part b) of Proposition 2.[29] $K$ captures the relative size of price risk and independent basis risk. Since $K$ decreases in $\operatorname{var}(\tilde{g})$ and increases in $\operatorname{var}(\tilde{P})$, $X^{vm,2}$ exhibits intuitively plausible comparative statics.

The optimal hedging position for a prudent decision maker under M.1 and under M.2 is an underhedging position. If the model is applied to corporate risk management, its predictions are qualitatively consistent with some stylized facts about corporate hedging as presented by Bodnar et al. (1998). Hence, basis risk offers an explanation for why firms on average only hedge a fraction of their perceived exposures. Other explanations are based on ‘selective hedging’ (Stulz, 1984), the existence of independent untradable quantity risk (Benninga et al., 1985) and correlation between price risk and quantity risk (Brown and Toft, 2002).

[28] Since $\operatorname{var}(\tilde{g}\tilde{P}) = \mathrm{E}[\tilde{P}^2]\operatorname{var}(\tilde{g})$ and $\operatorname{cov}(\tilde{P}, \tilde{g}\tilde{P}) = 0$, it is straightforward to show that $\operatorname{var}(\tilde{W}) = (Q - bX)^2 \operatorname{var}(\tilde{P}) + b^2 X^2\, \mathrm{E}[\tilde{P}^2]\operatorname{var}(\tilde{g})$. (6) follows directly.

[29] Under M.1, a prudent decision maker’s optimal forward position is always below the variance-minimizing position $X^{vm,1} = \beta\, Q$ . Under M.2, a similar statement cannot be derived.

# 5 An extension: Quantity risk

This section briefly presents an extended version of the model with untradable quantity risk in addition to price risk and basis risk of type M.1 or M.2. The decision maker's exposure is no longer deterministic at $Q$ but random, amounting to $Q\tilde{z}$. Quantity risk is modeled using a new random variable $\tilde{z}$ with support in the interval $[\underline{z}, \overline{z}]$ where $0 \leq \underline{z} < 1 < \overline{z} < \infty$ and $\mathrm{E}[\tilde{z}] = 1$. $\tilde{z}$ is stochastically independent of all other random variables. Given quantity risk, final wealth becomes $\tilde{W} = \tilde{P}Q\tilde{z} + (f - \tilde{F})X$.[30]

Under M.1, final wealth reads

$$\tilde{W} \;=\; \Big(\tilde{z} - 1\Big)\Big(\alpha + \beta\tilde{F}\Big)Q + \beta Q\tilde{F}\tilde{\gamma}\tilde{z} + \xi(\tilde{F}, X)\,, \qquad (7)$$

where $\xi(\tilde{F}, X)$ is defined as in (2). The first two summands on the RHS of (7) capture the effect of quantity risk whereas the second contains the basis risk. All three summands are closely related through $\tilde{F}$.

According to Proposition 1, basis risk of type M.1 causes a prudent decision maker to underhedge against price risk. Benninga et al. (1985) show that, absent basis risk, untradable quantity risk causes the same. The following corollary shows that underhedging is still optimal when the decision maker is simultaneously exposed to basis risk of type M.1 and quantity risk.

**Corollary 1** *Suppose that the forward market is unbiased, M.1 holds and there is independent quantity risk. The optimal forward position is a short hedge. An expected underhedging position, $X^* < \beta Q$, is optimal if and only if the decision maker is prudent.*[31]

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[30] Absent basis risk, Benninga et al. (1985) analyze both untradable as well as tradable quantity risk.
[31] The Appendix also provides a proof that $X^* >\ [=]\ \beta Q$ if and only if $U'''(\cdot) <\ [=]\ 0$.

In order to gain more insight into how basis risk and quantity risk interact, it is useful to start at the expected full hedge, $X = \beta Q$, where $\xi(\tilde{F}, X)$ is a constant.[32] Assuming a prudent decision maker, very low realizations of final wealth attract particular attention. Suppose that $z < 1$. The first term on the RHS of (7) is negative and decreases in $F$, creating an incentive to underhedge that does not exist in the absence of quantity risk. $z < 1$ also implies that the second term on the RHS of (7) becomes smaller (in absolute terms) such that the incentive to underhedge which it represents is smaller than in (2). Now, consider $z > 1$. The first term is positive and increasing in $F$, establishing a motive to *over*hedge. The second term is larger (in absolute terms) such that the incentive to underhedge grows. Hence, it is neither obvious for $z < 1$ nor for $z > 1$ what exactly the impact of quantity risk on the optimal forward position is, given the current set of assumptions.[33] Fortunately, Corollary 1 shows that quantity risk does not alter the result of Proposition 1.

Under M.2 and quantity risk, final wealth is given by $\tilde{W} = \tilde{P}(Q\tilde{z} - bX) - \tilde{g}\tilde{P}(bX) + X(f - a)$. The variance-minimizing forward position is the same as in (6) since quantity risk is independent and untradable. The following result shows that quantity risk does not alter the result of Proposition 2:

**Corollary 2** *Suppose that the forward market is unbiased, M.2. holds and there is independent quantity risk. The optimal forward position is a short hedge. It is an expected underhedge, $X^* < Q/b$, provided that $U'''(\cdot) \geq 0$.*

[32] As in the absence of quantity risk, this is the variance-minimizing forward position. To see why, notice that $\mathrm{var}(\tilde{W}) = Q^2\,\mathrm{var}(\tilde{P}\tilde{z}) + X(X - 2\beta Q)\,\mathrm{var}(\tilde{F})$ since independence of $\tilde{z}$ from $\tilde{F}$ and $\tilde{\gamma}$ and conditional independence of $\tilde{\gamma}$ from $\tilde{F}$ imply $\mathrm{cov}(\tilde{P}\tilde{z}, \tilde{F}) = \beta\,\mathrm{var}(\tilde{F})$.

[33] If final wealth is expressed as $\tilde{W} = \tilde{P}Q\tilde{z} + (f - \tilde{F})X$, it becomes apparent that introducing quantity risk $\tilde{z}$ increases the riskiness of price risk $\tilde{P}$ in the sense of Rothschild and Stiglitz (1970). Hence, our problem is isomorphic to the problem of choosing an optimal exposure towards $\tilde{F}$ in the presence of a background risk that is dependent and has undergone an increase in risk. A statement on how $X^*$ changes as a result of this increase in risk would require additional assumptions on either the utility function or the distributions of $\tilde{P}$ and $\tilde{\gamma}$ or both. See Athey (2002) and Gollier (2001, Ch. 8).

The interaction between basis risk and quantity risk under M.2 can be analyzed using

$$\tilde{W}\Big(X = Q/b\Big) \;=\; \frac{Q}{b}\left[\frac{[\tilde{z} - (1+\tilde{g})](\tilde{F} - a)}{1+\tilde{g}} + f - a\right]. \tag{8}$$

As this analysis is basically the same as for M.1, it can be left to the reader.

## 6 Conclusions

This paper analyzes optimal cross hedging with forward contracts. It focuses on two multiplicative relationships between basis risk on the one hand and the spot price or the forward price on the other. These relationships imply that basis risk is proportional to the level of the spot price or the forward price. The paper shows that the optimal forward position under these multiplicative specifications crucially depends on the decision maker's absolute prudence. If basis risk is multiplicatively combined with the forward price as under assumption M.1, there is a direct link between the decision maker's absolute prudence and the optimal forward position in an unbiased forward market: Underhedging is optimal if and only if absolute prudence is positive. Under assumption M.2 where basis risk is multiplicatively combined with the spot price, non-negative absolute prudence is a sufficient condition for the optimality of an underhedging position. As an extension shows, these results are robust to the introduction of quantity risk. In sum, the paper shows how, given multiplicative basis risk, the decision maker's preferences solve the quintessential tradeoff between reducing price risk and creating basis risk.

The results derived here are closely related to two other standard decision problems under risk, one being a portfolio problem, the other being an insurance problem. The analogous portfolio problem is one in which the decision maker starts from a position in one asset and has to decide on how much of another asset to add to the existing position given that the other asset provides no excess return. In the analogous insurance problem, the insurer can only observe the realized loss with an error such that the indemnity does not perfectly match the loss.[34]

Moschini and Lapan (1995) and Brown and Toft (2002) show that there is a hedging role for options if quantity risk and price risk are jointly (log-) normally distributed. As Wong (2003) demonstrates, there is a role for options even if quantity risk and price risk are independent. It would be interesting to see whether price risk and multiplicative basis risk create a non-linearity in the decision problem that leads to similar results. However, this is left for future research.

[34]Gollier (1996) analyzes the nature of the optimal insurance contract in such an environment. He shows that a disappearing deductible is optimal if the policyholder is prudent and that this deductible equals zero if insurance is costless.

# Appendix

As Propositions 1 and 2 are special cases of Corollaries 1 and 2, the respective corollary is proven first.

## Proof of Corollary 1

The equivalence of $X^* < \beta Q$ to $U'''(\cdot) > 0$ is proven first. Unbiasedness and the law of iterated expectations imply that the first-order condition in (1) can be written as $-\operatorname{cov}(\mathrm{E}[U'(\tilde{W}^*)|F], \tilde{F}) = 0$ where $\tilde{W}^* = (\alpha + \beta\tilde{F}(1+\tilde{\gamma}))Q\tilde{z} + (f - F)X^*$. Hence, $\mathrm{E}[U'(\tilde{W}^*)|F]$ is either a constant or it is decreasing in some interval of $F$ while increasing in some other interval of $F$. This requires

$$\begin{aligned}
\frac{\partial\,\mathrm{E}\Big[U'(W)\Big|F\Big]}{\partial F} &= \mathrm{E}\Big[U''(\tilde{W})\Big(\beta(1+\tilde{\gamma})Q\tilde{z} - X\Big)\Big|F\Big] \\
&= \mathrm{E}\Big[U''(\tilde{W})(\beta Q\tilde{z} - X)\Big|F\Big] + \beta Q\,\mathrm{E}\Big[U''(\tilde{W})\tilde{z}\tilde{\gamma}\Big|F\Big] \\
&= \Big(\beta Q - X\Big)\,\mathrm{E}\Big[U''(\tilde{W})\Big|F\Big] + \beta Q\operatorname{cov}\Big(\mathrm{E}[U''(\tilde{W})|z], \tilde{z}\Big|F\Big) \\
&\quad + \beta Q\,\mathrm{E}\Big[\tilde{z}\operatorname{cov}\Big(U''(\tilde{W}), \tilde{\gamma}\Big)\Big|F\Big]
\end{aligned} \tag{9}$$

to be equal to zero everywhere or to vary in sign. The derivation of (9) uses the fact that $\tilde{z}$ is independent of $\tilde{\gamma}$ and $\tilde{F}$, implying $\mathrm{E}[\tilde{\gamma}|z] = \mathrm{E}[\tilde{\gamma}] = 0$ for all $F$. The law of iterated expectations is used to replace $\mathrm{E}[U''(\tilde{W})\tilde{z}\tilde{\gamma}|F]$ by $\mathrm{E}[\tilde{z}\,\{\mathrm{E}[\tilde{\gamma}|z]\,\mathrm{E}[U''(\tilde{W})|z] + \operatorname{cov}(U''(\tilde{W}), \tilde{\gamma}|z)\}|F] = \mathrm{E}[\tilde{z}\operatorname{cov}(U''(\tilde{W}), \tilde{\gamma})|F]$.

In order to sign the first covariance in (9), consider $\partial\mathrm{E}[U''(\tilde{W})|z]/\partial z = \mathrm{E}[U'''(\tilde{W})\,(\alpha + \beta\tilde{F}(1+\tilde{\gamma}))Q|z] = \mathrm{E}[U'''(\tilde{W})\tilde{P}Q|z]$. The fact that $P > 0$ for all $F$, $\gamma$, $z$ and $Q > 0$ implies $\operatorname{sgn}\operatorname{cov}(\mathrm{E}[U''(\tilde{W})|z], \tilde{z}|F) = \operatorname{sgn} U'''(\cdot)$ for all $F$. The sign of $U'''(\cdot)$ also determines the sign of the other covariance in (9): Since $\partial U''(W)/\partial\gamma = U'''(W)\beta FQz$, the fact that $\beta$, $Q$, $F > 0$ and $z \geq 0$ (with $z > 0$ in at least some states) implies $\operatorname{sgn}\operatorname{cov}(U''(\tilde{W}), \tilde{\gamma}) = \operatorname{sgn} U'''(\cdot)$ for all $F$.

The remainder of the proof is by contradiction. Suppose that $U'''(\cdot) > 0$. Since $\beta$, $Q > 0$ and $z \geq 0$ (with $z > 0$ in at least some states), the second and third summand on the RHS of (9) are positive. If $(\beta Q - X) \leq 0$, the first summand will be non-negative as well due to risk aversion such that there is no interval in which $\mathrm{E}[U'(\tilde{W})|F]$ does not increase in $F$. Since this leads to a contradiction, the optimum must be characterized by $(\beta Q - X^*) > 0$ for $U'''(\cdot) > 0$. (Using the same argument, it is straightforward to show that $U'''(\cdot) < [=]\, 0$ is equivalent to $(\beta Q - X^*) < [=]\, 0$.)

In order to prove that $X^* > 0$, irrespective of the sign of $U'''(\cdot)$, evaluate (9) at $X = 0$. As the first line of (9) shows, the expression is negative since $U''(\cdot) < 0$, $\beta$, $Q$, $(1+\gamma) > 0$ and $z \geq 0$ (with $z > 0$ in at least some states). Hence, the first-order condition in (1), evaluated at $X = 0$, is positive. $X^* > 0$ then follows from the concavity of the problem. □

## Proof of Proposition 1

Without quantity risk, $z = 1$ in all states. Hence, the first covariance in (9) equals zero. Otherwise, the proof of Corollary 1 applies. □

## Proof of Corollary 2

Using the law of iterated expectations, the notation can be simplified by defining

$$A(X) = \mathrm{cov}\Big(U''(\tilde{W}(X)), \tilde{g}\Big|P\Big) = \mathrm{cov}\Big(\mathrm{E}\Big[U''(\tilde{W}(X))\Big|g\Big], \tilde{g}\Big|P\Big), \tag{10}$$

$$B(X) = \mathrm{cov}\Big(U''(\tilde{W}(X)), \tilde{z}\Big|P\Big) = \mathrm{cov}\Big(\mathrm{E}\Big[U''(\tilde{W}(X))\Big|z\Big], \tilde{z}\Big|P\Big), \tag{11}$$

$$C(X) = \mathrm{cov}\Big(U'(\tilde{W}(X)), \tilde{P}\Big) = \mathrm{cov}\Big(\mathrm{E}\Big[U'(\tilde{W}(X))\Big|P\Big], \tilde{P}\Big), \tag{12}$$

$$D(X) = \mathrm{cov}\Big(U'(\tilde{W}(X))\,\tilde{P}, \tilde{g}\Big) = \mathrm{cov}\Big(\mathrm{E}\Big[U'(\tilde{W}(X))\,\tilde{P}\Big|g\Big], \tilde{g}\Big). \tag{13}$$

Using M.2, $\mathrm{E}[\tilde{g}] = 0$ and $f = \mathrm{E}[\tilde{F}] = a + b\mathrm{E}[\tilde{P}]$ due to unbiasedness, one can rewrite the LHS of the first-order condition in (1) as

$$
\begin{aligned}
& \mathrm{E}\Big[U'(\tilde W(X^*))\,\Big(f-(a+b\tilde P(1+\tilde g))\Big)\Big] \\
= \;& \mathrm{E}\Big[U'(\tilde W(X^*))\,\Big(f-(a+b\,\tilde P)\Big)\Big]-b\,\mathrm{E}\Big[U'(\tilde W(X^*))\,\tilde g\tilde P\Big] \qquad (14)\\
= \;& -b\Big[C(X^*)+D(X^*)\Big]\,.
\end{aligned}
$$

(1) and (14) imply $[C(X^*)+D(X^*)]=0$ since $b>0$.

The remainder of the proof is based on signing $A(X)$ to $D(X)$. In order to sign $A(X)$, it is useful to derive

$$
\frac{\partial\,\mathrm{E}\Big[U''(\tilde W)\Big|g,P\Big]}{\partial\,g} \;=\; -\,bX\,\mathrm{E}\Big[U'''(\tilde W)\,P\Big|g,P\Big] \qquad \forall\,g,P\,. \tag{15}
$$

Since $b$, $P>0$, (15) implies $\operatorname{sgn} A(X)=-\operatorname{sgn}\{XU'''(\cdot)\}$. The sign of $B(X)$ can be determined using

$$
\frac{\partial\,\mathrm{E}\Big[U''(\tilde W)\Big|z,P\Big]}{\partial\,z} \;=\; Q\,\mathrm{E}\Big[U'''(\tilde W)\,P\Big|z,P\Big] \qquad \forall\,z,P\,. \tag{16}
$$

Together with $P$, $Q>0$, (16) implies $\operatorname{sgn} B(X)=\operatorname{sgn} U'''(\cdot)$. To sign $C(X)$, notice that

$$
\begin{aligned}
\frac{\partial\,\mathrm{E}\Big[U'(\tilde W)\Big|P\Big]}{\partial\,P} \;&=\; \mathrm{E}\Big[U''(\tilde W)\,\Big(Q\tilde z-bX(1+\tilde g)\Big)\Big|P\Big] \\
&=\; \Big(Q-bX\Big)\mathrm{E}\Big[U''(\tilde W)\Big|P\Big]-bX\,A(X)+Q\,B(X) \qquad \forall\,P
\end{aligned} \tag{17}
$$

since $\mathrm{E}[\tilde e]=0$ and $\mathrm{E}[\tilde z]=1$. Signing $D(X)$ uses the fact that

$$
\frac{\partial\,\mathrm{E}\Big[U'(\tilde W)\,\tilde P\Big|g\Big]}{\partial\,g} \;=\; -\,bX\,\mathrm{E}\Big[U''(\tilde W)\,\tilde P^2\Big|g\Big] \qquad \forall\,g\,. \tag{18}
$$

Hence, $U''(W)<0$ and $b>0$ imply $\operatorname{sgn} D(X)=\operatorname{sgn} X$.

Consider $X=0$. Then, (17) reduces to $\partial\,\mathrm{E}[U'(\tilde W)|P]/\partial\,P=\mathrm{E}[U''(\tilde W)\,Q\tilde z|P]$. Since $U''(\cdot)<0$, $Q>0$ and $z\geq 0$ (with $z>0$ in at least some states), the conditional expectation is negative for all $P$ such that $C(0)<0$. In addition, $D(0)=0$. Hence, (14) is positive if evaluated at $X=0$. The concavity of the problem implies $X^*>0$. This proves part a).

Now, consider $X = Q/b$. Suppose that $U'''(W) \geq 0$ for all $W$. $B(Q/b) \geq 0$ and $D(Q/b) > 0$ follow directly. Also, $Q > 0$ implies $A(Q/b) \leq 0$. Hence, $C(Q/b) \geq 0$ by (17). Taken together, (14) is negative at $X = Q/b$. Hence, $X^* < Q/b$ due to the concavity of the problem. This proves part b). □

### Proof of Proposition 2

Without quantity risk, $z = 1$ in all states. Hence, $B(X) = 0$ for all $X$ such that (17) simplifies slightly. Otherwise, the proof remains the same as that of Corollary 2. □

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