Debt allocation: To fix or float?

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March 15, 2006

Abstract

The question whether a housebuyer should choose fixed or adjustable interest rate on his mortgage has been analyzed extensively in the literature. Here we ask a different question: Which fraction of the house financing should be to the fixed rate and which share should be to the floating rate? The question is already relevant for mortgage choices in several countries and will probably become relevant for more mortgage markets due to the ongoing liberalization of such markets.

Our economic agent has an exogenously determined initial amount of debt. The agent is equipped with a constant relative risk aversion utility function and a deterministic terminal wealth before debt interest payments and faces the described debt allocation problem. By this formulation the problem is related to the seminal Merton (1969), Merton (1971) asset allocation problem. In order to model fixed and floating interest rates we use a version of the Hull and White (1990) term structure model, essentially the Vasicek (1977) model fitted to the initial term structure.

First, the static case is considered, where no rebalancing of debt is allowed after the initial point in time. The optimal fraction of floating rate debt is shown to be proportional to the wealth-to-debt ratio. No closed form solution for the optimal fraction of floating rate debt is available, but some analytical approximations are discussed and compared with numerical solutions. Next, the dynamic case is treated where the debt portfolio can be rebalanced continuously at no cost. In this case closed form solutions for the optimal fraction of floating rate debt are available We find a

*Earlier versions of this paper have been presented at *Workshop on finance and insurance*, June 2001, Stockholm, Sweden, *FIBE*, January 2002, NHH, Bergen, *3rd Bachelier conference*, Crete, June 2002, Norwegian School of Economics, Bergen, November 2002, *2nd Annual Conference on Personal Risk Management*, Toronto, November 2002, *Symposium on Asset Allocation and Pension Management*, Copenhagen, November 2004. Comments and suggestions from Michael Brennan, Steinar Ekern, Jørgen Haug, Steen Koekebakker, Moshe Milevsky, Kristian Miltersen, Gordon S. Roberts, and Chester Spatt are most appreciated.

surprisingly low increase in welfare, measured by expected utility, in the dynamic case compared to the static case. The optimal debt portfolio in the dynamic case is sensitive to the initial shape of the initial forward interest rates and therefore may or may not resemble the static case.

JEL-classification: D14, G11, D81, D91, E43.

1 Introduction

Household finance[1], by analogy with corporate finance, asks how households use financial instruments to attain their objectives. One can distinguish between *positive household finance* - what households actually do, and *normative household finance* - what households should do. One of the largest household investments is, for most people, the purchase of a house. The mortgage contract is therefore the most important financial contract for a typical household.

There exist two fundamentally different mortgages: Floating rate mortgages (also called adjustable rate mortgages - ARM) and fixed rate mortgages (FRM). The fixed rate mortgage specifies a horizon over which the interest rate is fixed. These two contracts may be equipped with additional options. An important example for FRM is the prepayment at par option . That is, the borrower can at any time call the mortgage at par value, it is not possible to owe more to the lender than the original face value. ARM's can for example be equipped with a cap on the interest rate, i.e., the floating interest rate can not exceed the cap.

Both the menu of available mortgages and the distribution of actually used mortgages vary from country to country. In Norway, Great Britain, and Sweden more than 40% of the mortgages are adjustable, see European Central Bank 2003; Miles (2005). But also Austria, Italy, Ireland, and the Netherlands have more than 20% adjustable mortgages. Only the US, France, and the Netherlands have more than 60% fixed for 10 years or more mortgages (including only European countries and the US). In Ireland and Spain more than 60% of the mortgages are fixed for less than 5 years. This information also emphasize that whereas in some countries the horizon of the FRMs are 20 or 30 years, in other markets the horizon maybe shorter than 5 years. Traditionally, in the American market FRM's are issued with a prepayment-at-par option and financing of a house has been done either by an FRM or an ARM for typically a long horizon of 20 years or more. The American mortgage market has been analyzed extensively in the academic literature. See e.g., Dunn and Spatt (1985, 1988, 1999), Campbell and Cocco (2003).

The following example illustrates mortgage alternatives in the Norwegian market. Most Norwegian banks offer customers a menu of choices for mortgage financing. The interest rate conditions of the major Norwegian bank Postbanken (part of the DnB group) are presented in table (1). The conditions depend on whether the loan amount is within 60% or 80% of the value of the house

[1] a reserch field that have attracted much recent attention and was the topic of John Campbell's Presidential Address to the American Finance Association in January 2006, see Campbell (2006).

Oct 30, 2002	floating $> \frac{1}{2}$m	$< \frac{1}{2}$m	fixed 3 y	5 y	10 y
$< 60\%$	8.1%	8.45%	7.45%	7.40%	7.40%
$< 80\%$	8.85%	9.15%	7.9%	7.85%	7.85%
Mar 6, 2006	$> \frac{1}{2}$m	$< \frac{1}{2}$m	3 y	5 y	10 y
$< 60\%$	3.35%	3.80%	4.40%	4.65%	4.90%
$< 80\%$	4.20%	4.50%	4.85%	5.10%	5.35%

Table 1: Interest rate conditions of Postbanken Oct 30, 2002 and March 6, 2006, found at `www.postbanken.no`.

and whether the amount of loan is above or below 500 000 NOK (roughly 70 000 USD). The customer can choose between floating, fixed for 3, 5, or, 10 years interest rates, and also the allocation of debt between these 4 alternatives. Observe that the fixed rate alternatives do not include any prepayment option. If the interest rates decrease during the term of the contracts, customer may end up owing the lender more money than the original loan amount. In medias advices of the kind "Fix everything now!", "Let your debt float!" flourish. Sometimes you also see more refined advices as: "Keep half your debt fixed and let the other part float!".

Motivated by this example, we study an economic agent who borrows an exogenously determined amount of money. Two types of loan alternatives are present: Loan with a fixed interest rate through the loan horizon and loans with floating interest rates. Our model is simpler than the example as only one fixed rate debt alternative is present. By finding the optimal floating rate fraction, we also residually determine the optimal fraction of fixed rate debt. Observe that this question is different from the question addressed elsewhere in the literature (see Campbell and Cocco (2003)): Under which conditions should an agent choose ARM or FRM? We do not include any prepayment option.

Our model does not include labour income, consumption, taxes, or inflation. One reason for this is to maintain parsimony. Another reason is that for many markets the horizon of the fixed rate alternative may be realitive short (1 to 5 years). It is reasonable that labour income, consumption and inflation have less impact for such short horizons, than for horizons of, say, 20-30 years.

Interest rates are driven by a one factor model. In a setting with several fixed rate alternatives with different horizons, one implication of the one-factor model is that only one fixed rate alternative will be chosen at any time. A natural extension of this work is to include more factors. In a multi-factor model the optimal debt portfolio may include several fixed rate debt alternatives with different horizon.

We assume that both fixed and floating rate interest rates are determined from the prevailing term structure of interest rates derived from a financial market. There is no mark-up or spread added to the mortgage rates compared to the observed interest rates.

The investor must determine his initial distribution between fixed rate and floating rate loan. Two extreme cases are treated with respect to intermediate rebalancing of the loan portfolio: No rebalancing and continuous rebalancing.

The basic intuition of our model is as follows: The more risk averse the agent is, the larger fraction of his debt he wants to the fixed rate. Fixed rate debt thus serves as an insurance against future fluctuating interest rates.

The problem is in many ways related to the classical Merton (1969), Merton (1971) problem. Merton studies the investment decision or asset allocation where capital may be invested either in a risky security or to the riskfree interest rate. Whereas Merton deals with allocation of assets, we focus on the liability side of the balance sheet and study debt allocation. In our set-up fixed rate debt corresponds to a 'risky' investment in the sense that the intermediate market value of the fixed-rate debt fluctuates randomly. Although we allow for stochastic interest rates, floating rate debt has similar dynamics as a bank or money market account which in Merton's model corresponds to the 'riskfree' investment.

A term structure model including random interest rates is essential to our problem. In order to keep things simple we use a one factor Gaussian spot interest rate process with mean reversion. The same process was first used by Vasicek (1977). By assuming a special structure of the market price of interest rate risk process the Vasicek model is consistent with the initial observable term structure. This extension is credited Hull and White (1990). By calibrating the model to the initial observable term structure one does not need to make ad-hoc assumption with respect to the market price of interest rate risk (like the typical example of a constant market price of interest rate risk). At the same time forward interest rates, which are observable, in our model as well as in real world financial markets, enter the model in a natural way.

Our set-up is similar to the recent literature on asset allocation in models with stochastic interest rates (Sørensen, 1999; Brennan and Xia, 2000; Bajeux-Besnainou, Jordan, and Portait, 2001; Munk and Sørensen, 2001). Brennan and Xia (2000) and Bajeux-Besnainou et al. (2001) suggest a solution to the apparent asset allocation puzzle established by Canner, Mankiw, and Weil (1997), However, most of these papers assume a constant market price of risk. This difference may play a crucial role when it comes to optimal allocations.

Section 2 of the article contains a description of one version of the Hull and White (1990) term structure model.

Section 3 describes the agent's problem.

In section 4 we present results for a static model without the possibility to rebalance the loan portfolio before expiration. We derive a lower bound for the fixed rate which is interpreted as follows: The agent will not borrow at the floating rate (though he may wish to *lend* money at the floating rate) if the fixed rate is below this lower bound. This lower bound depends only on, in a specific sense, expected future interest rates. We also derive an upper bound for the fixed rate interpretable as follows: The agent will not borrow at the fixed rate (also for this case he may wish to lend money to the fixed rate) if the fixed interest rate is above this upper bound. In addition to expectations about the future interest rates this upper bound depends on characteristics of the agent related to his financial wealth and preferences. For values of the fixed rate between the lower and upper bounds it is optimal to keep positive fractions of both floating rate debt and fixed rate debt. We show that the optimal fraction of floating rate debt is proportional to the initial wealth-to-debt ratio. No closed form solutions for the optimal fraction of floating rate are available, but we discuss briefly 3 different approximations and compare them briefly with numerical solutions.

In section 5 the agent is allowed to rebalance his loan portfolio continuously at no cost. For this problem we derive closed form expressions both for optimal expected utility and optimal fractions of floating rate debt. We present some numerical comparisons of the dynamic case with the static case both in terms of welfare measured by optimal expected utility and in terms of initial fractions of floating rate debt. Our numerical examples indicate, perhaps surprisingly, only a marginal increase in optimal expected utility. It turns out that the interplay between the market price of risk and the initial forward rates plays an important role, especially for the optimal debt fractions. Miles (2005) also discusses how initial yields may affect the consumer's choice of mortgage. As a consequence the optimal initial fractions of debt may be substantially different from the static case. Several examples are included to illustrate this point.

Finally, section 6 concludes the article.

2 The agent's problem

We denote the initial point in time by s and the horizon by T.

We assume that utility is derived from final time T wealth only, and that the agent is equipped with a constant relative risk aversion (CRRA) utility function given by

$$u(x) = \frac{1}{1-\rho} x^{1-\rho}, \tag{1}$$

where, as usual, ρ can be interpreted as the relative risk aversion coefficient $(\frac{-u''(x)}{u'(x)} x = \rho)$. Here ρ is assumed positive and the special case $\rho = 1$ corresponds to the utility function $u(x) = \ln(x)$.

The floating rate loan is assumed to accrue interest according to a continuously compounded spot rate r_t. The accrued floating rate over the interval (t, T) is given by

$$R_t = \int_t^T r_u du,$$

where the dependence on T is suppressed. We denote by R the initial time s accrued (s, T) interest rate. In the first case (the static case) we consider below R is denoted by $\tilde{R}$ since this is the only random variable in question.

Denote by $f_t(u)$ the continuously compounded forward rate for time u observable at time t, $s \leq t \leq u \leq T$. The fixed loan rate for the period (t, T) is determined at each point in time from the time t observable forward rates as

$$r_t^x = \frac{1}{T-t}\int_t^T f_t(u)du.$$

The initial time s fixed rate is denoted by r^x, i.e., $r^x = r_s^x$. The connection between the fixed rate/forward rates and the market price of a default free unit discount bond with expiration T, $P_{t,T}$, is

$$P_{t,T} = e^{-\int_t^T f_t(u)du} = e^{-r_t^x(T-t)}.$$

The initial time s amount of debt is exogeneously given by the constant D. The time t market value of debt is denoted by D_t for $t \leq T$.

Furthermore, we assume that the agent has a deterministic and exogenously given time T wealth $\bar{W}$. This constant can be interpreted as the total wealth before debt payments at time T, and in particular, includes potential net savings in the period (s, T). Typically, a significant share of $\bar{W}$ will serve as the *collateral* for the loan, e.g. in the case of house financing, the time T value of the acquired house is included in $\bar{W}$. We therefore refer to $\bar{W}$ as the collateral.

Finally, all interest payments are assumed to take place at the horizon T.

Below some notation is introduced. The time t market value of wealth W_t is simply the difference between the time t market value of the collateral and the debt, i.e.,

$$W_t = \bar{W}e^{-r_t^x(T-t)} - D_t, \tag{2}$$

In particular, the initial time s market value of time T wealth is denoted by W, i.e., $W = W_s$.

We denote the ratio between the time t market value of wealth and the (market value of) the time t debt by L_t. Then

$$L_t = \frac{W_t}{D_t} = \frac{\bar{W}e^{-r_t^x(T-t)} - D_t}{D_t}.$$

Sometimes we refer to L_t as the (time t) *wealth to debt ratio*. According to our practice we denote the initial time s wealth to debt ratio by L, that is $L = L_s$.

3 The static case: No intermediate rebalancing of debt

In this section we assume that no rebalancing of the debt portfolio can take place after the investor has chosen the initial distribution between fixed and floating rate debt.

We denote the fraction of floating rate debt to total debt by α (the optimal value of α is denoted by α^*). The agent's terminal (time T) wealth W_T is the difference between the exogeneous time T wealth and the sum of the floating and fixed rate debt payments, where the debt payments include both the return of the face values (principals) and payments of interest rates. This may be written as

$$W_T = \bar{W} - \alpha D e^{\tilde{R}} - (1-\alpha) D e^{r^x(T-s)}.$$

Observe that $\tilde{R}$ is the only random variable in this expression. Unless $\tilde{R} = \int_s^T r_t dt$ is bounded there may be a potential mathematical problem of negative terminal wealth for high values of α. The case of negative terminal wealth has a clear economic interpretation as *bankruptcy*, but the utility function is not defined in this case.

Values of $\alpha < 0$ represent *short* positions of floating rate debt, which means that the investor acts as *lender* instead of *borrower*. Similarly, values of $\alpha > 1$ imply *short* positions of fixed rate debt, which means that the investor acts as a bond investor instead of a bond issuer. We do not formally exclude values of α greater than 1 or lower than zero, but since we are primarily concerned with optimal debt allocation, we focus on the case $0 < \alpha < 1$ in numerical examples.

Now rewrite[2] W_T in terms of the time s wealth-to-debt ratio L as

$$W_T = DLe^{r^x(T-s)} \left[1 + \frac{\alpha}{L}\left(1 - e^{\tilde{R} - r^x(T-s)}\right)\right].$$

The agent's problem is stated as

$$\max_{\alpha} E\left[u(W_T)\right].$$

Differentiating yields

$$\frac{\partial}{\partial \alpha} = DLe^{r^x(T-s)} E\left[u'(W_T)(1 - e^{\tilde{R} - r^x(T-s))})\right].$$

The first order condition of this problem is

$$E\left[u'(W_T)(1 - e^{\tilde{R} - r^x(T-s))})\right] = 0. \tag{3}$$

We now define the $\tilde{R}_\Delta$ as the difference between the floating rate and the fixed rate in the period (s, T), i.e., as

$$\tilde{R}_\Delta = \tilde{R} - r^x(T-s).$$

Also, for future use, let

$$\mu_\Delta = E[\tilde{R}_\Delta] = E[\tilde{R}] - r^x(T-s).$$

The first order condition may then be rewritten as

$$E\left[u'(W_T)(1 - e^{\tilde{R}_\Delta)})\right] = 0. \tag{4}$$

[2]Observe that the following expression is on the form $W_T = K + \alpha \tilde{Y}$, where K is constant and $\tilde{Y}$ is a random variable. The problem has thus the same structure as both the classical asset allocation problem, see e.g. Huang and Litzenberger (1988) as well as optimal purchase of insurance problems, cf. Mossin (1968), as e.g., explained in the textbook by Eeckhoudt and Gollier (1995).

3.1 A lower fixed rate bound for floating rate debt

In order to analyze this first order condition (3) further we apply the standard arguments used, e.g., in Huang and Litzenberger (1988). We set $\alpha = 0$. For this special case all debt is fixed rate debt and $W_T = \bar{W} - D\exp\left(r^x(T-s)\right)$ is deterministic. The first order condition may then be written as

$$LDu'(W_T)\left(e^{r^x(T-s)} - E\left[e^{\tilde{R}}\right]\right).$$

The value of the first order condition may be interpreted as the marginal increase in expected utility of time T wealth from a marginal increase in α. If the value of the above expression is positive, we may conclude that the optimal value of α is positive. We assume that $E[e^{\tilde{R}}]$ exists and can be written on the form

$$E[e^{\tilde{R}}] = e^{\mu}$$

for a constant μ.

Now, define

$$r_L = \frac{\mu}{T-s}. \tag{5}$$

These arguments lead to the following result:

Proposition 1 *The optimal fraction of floating rate debt α^* is strictly positive if and only if the fixed rate r^x is strictly greater than r_L defined in expression (5).*

This result is interpreted as follows: If $r^x > r_L$, it is optimal to accept *some* floating interest rate loan. If the fixed rate is r_L or lower, at least 100% of the loan amount is financed by fixed rate debt. In the case where strictly more than 100% of the loan amount is financed by fixed rate debt, the agent 'shorts' floating rate debt, i.e., the agent *lends* instead of *borrows* to the floating rate.

Note that this result holds for any utility function with strictly positive marginal utility and does therefore not depend on our particular choice of utility function in expression (1).

In the spirit of expression (4) this condition can equivalently be expressed as follows: The optimal fraction of floating rate debt $\alpha^* > 0$ if and only if $\ln E[e^{\tilde{R}_\triangle}] = \mu - r^x(T-s) < 0$. In order to interpret the latter condition observe that $\ln E[e^{\tilde{R}_\triangle}]$ is interpreted as the expected difference between the floating rate and the fixed rate in the period (s,T) (It is not correct e.g., to simply compare $E[\tilde{R}]$ with $r^x(T-s)$.).

It is natural to define the *risk premium* of floating rate loans compared to fixed rate loans as the expected gain by accepting floating rate loan instead of fixed rate loan, i.e., as $r^x(T-s) - \mu = -\ln E[e^{\tilde{R}_\triangle}]$. The reformulated condition then reads as follows: Accept *some* floating rate debt if and only if (1) the expected floating rate is lower than the fixed rate, or equivalently, (2) the risk premium is positive.

3.2 An upper bound for fixed rate debt

An upper bound for *some* fixed rate debt may be derived in an analogous matter. Define

$$\tilde{Z} = \frac{\bar{W}}{D} - e^{\tilde{R}}.$$

We now study the situation with only floating rate loan, i.e., we let $\alpha = 1$. By using the CRRA utility function in expression (1) the first order condition (3) is proportional to

$$E[\tilde{Z}^{1-\rho}] + \left(e^{r^x(T-s)} - \frac{\bar{W}}{D}\right) E[\tilde{Z}^{-\rho}]$$

If this first order condition takes a negative value, it is optimal to decrease α, i.e., to accept *some* fixed rate loan. Define

$$r_U = \frac{1}{T-s}\left[\ln\left(\frac{\bar{W}}{D} - \frac{E[\tilde{Z}^{1-\rho}]}{E[\tilde{Z}^{-\rho}]}\right)\right]. \tag{6}$$

We now have the following result:

Proposition 2 *The optimal fraction of floating rate debt α^* is strictly less than 1 if and only if the fixed rate $r^x < r_U$ defined in expression (6).*

This result tells us that for fixed rates lower than r_U it is optimal to accept *some* fixed rate loan. As opposed to the lower bound the upper bound r_U depends on the agent specific factors, $\frac{\bar{W}}{D}$, ρ, and our specific choice of utility function (see expression (1)). Numerical results require the calculation of two moments of the random variable Z defined above. The expression for r_U also holds for the case $\rho = 1$, i.e., for $u(x) = \ln(x)$.

From expression (6) the following result follows immediately for the special case of a risk neutral agent:

Proposition 3 *For a risk neutral agent, i.e., $\rho = 0$, the upper bound equals the lower bound $r_U = r_L$.*

Thus, a risk neutral agent chooses the debt alternative with the lowest expected interest rate, i.e., either fixed rate loan or floating rate loan, never a combination of both.

3.3 Constant relative risk aversion? A reformulation

By inserting the assumed CRRA utility function (1) into the first order condition (4) we obtain

$$E\left[\left(1 + \frac{\alpha}{L}(1 - e^{\tilde{R}_\Delta})\right)^{-\rho}\left(1 - e^{\tilde{R}_\Delta}\right)\right] = 0. \tag{7}$$

From this expression it is clear that α^* depends on the time s wealth to debt ratio L, in addition to ρ and properties of $\tilde{R}_\Delta$ for a fixed time horizon $T - s$. Here α^* does not, for example, depend on the levels of either $\bar{W}$ or D.

By inspection of equation (7) it is clear that the optimal α is proportional to the parameter L, i.e., if $\bar{\alpha}$ solves the equation for $\bar{L}$, $k\bar{\alpha}$ will solve the equation for $k\bar{L}$, for any constant k.

From this proportionality property we see that the fraction of floating rate debt increases with the wealth to debt ratio L. However, a well known property of CRRA utility (1) is that the total fraction of 'risky investments' is independent of wealth. In order to obtain results in this spirit we reformulate the problem as follows: We now express the amount of floating rate debt as a fraction β of the time s market value of the time T wealth as

$$W_T = (\bar{W} - De^{r^x(T-s)})\left(1 + \beta\left(1 - e^{\tilde{R}_\Delta}\right)\right).$$

The first order condition of this reformulation is

$$E\left[\left(1 + \beta(1 - e^{\tilde{R}_\Delta})\right)^{-\rho}\left(1 - e^{\tilde{R}_\Delta}\right)\right] = 0. \tag{8}$$

By inspection of the first order condition (8) it is clear that the optimal β (denoted by β^*) does neither depend on $\bar{W}$, D, nor L.

As mentioned, the reformulation only involves a change of base for a fraction, so the total optimal *amounts* of floating rate debt are the same for the two formulations. Thus

$$D\alpha^* = (\bar{W}e^{-r^x(T-s)} - D)\beta^*.$$

This insight leads to a simpler way of calculating α^* for different wealth to debt ratios: First, calculate β^* which is independent of L. Then, calculate the corresponding α^* as

$$\alpha^* = L\beta^*,$$

i.e., the optimal α is given as the optimal β multiplied by the time s wealth to debt ratio.

3.4 Numerical illustrations — static case

Our numerical analysis is based on the first order condition (8) which can not be solved in closed form. We therefore solve it numerically and compare with 3 different approximations.

We assume that $\tilde{R}$ is normally distributed with expectation $\mu_{s,T}$ and variance $\sigma^2_{s,T}$, and use the more compact notation $\tilde{R} \sim N(\mu_{s,T}, \sigma^2_{s,T})$ for this. By imposing this assumption directly on $\tilde{R}$ we do not need to assume anything about the short term interest rate process. However, our assumed parameter values will be consistent with the short term interest rate process we apply for our numerical analysis in the continuous model later in the paper.

The first approximation is the Campbell and Viceira (2002) approximation adapted to our problem:

$$\hat{\beta}_{CV} = \frac{-\mu + r^x(T-s)}{\rho\sigma^2_{s,T}}. \tag{9}$$

This expression can equivalently be written as $\hat{\beta}_{CV} = \frac{-\ln E[e^{\tilde{R}_\Delta}]}{\rho\sigma^2_{s,T}}$, i.e., as the risk premium divided by the product of the relativ risk aversion coefficient and a variance. In addition to its simple structure and obvious relationship to optimal asset allocation in mean variance models, Campbell and Viceira (2002) show that it converges to the exact solution in the limit as the time horizon approaches zero for the asset allocation problem they study. They therefore conclude that their approximation performs better for short horizon problems than for long horizon problems.

One critical step in their approximation consists of (rewritten in our notation) replacing $(\tilde{R} - r^x(T-s))^2$ by $\sigma^2_{s,T}$. Obviously, numerical accuracy can be improved by realizing that $(\tilde{R}-r^x(T-s))^2 = ((\tilde{R}-\mu_{s,T})+(\mu_{s,T}-r^x(T-s)))^2$, and replacing the latter expression by its expectation $\sigma^2_{s,T} + (\mu^\Delta)^2$. This refinement leads to the following approximation

$$\hat{\beta}_{CV+} = -\frac{\mu - r^x(T-s) + \frac{1}{2}(\mu^\Delta)^2}{\rho\sigma^2_{s,T} + (\mu^\Delta)^2}. \tag{10}$$

Both the two previous approximations are best suited for shorter maturities. Below we present a third approximation based directly on the first order condition (8) (details in Appendix B), which should perform better than the two previous ones for long maturities.

$$\hat{\beta} = \frac{\sigma^2\rho - 2(1-M)^2 - \sigma\sqrt{\rho^2\sigma^2 - 2(1-M)^2(1+\rho)\rho}}{2(1-M)^3 + \sigma^2(1-M)\rho(\rho-1)}, \tag{11}$$

where

$$\sigma^2 = e^{2\mu^\Delta + \sigma^2_{s,T}}(e^{\sigma^2_{s,T}} - 1)$$

and

$$M = e^{\mu^\Delta + \frac{1}{2}\sigma^2_{s,T}}.$$

Table (2) presents the base case parameters, which are intended to be within reasonable ranges. In particular, the base case values of the mean reversion speed q and the volatility v are close to the values estimated for the Vasicek spot rate interest by Chan, Karolyi, Longstaff, and Sanders (1992). The chosen value of the interest rate level r is similar to the value used e.g., by Munk and Sørensen (2001), although the optimal *allocation* is not dependent on this parameter. The chosen time horizon represents a typical option in the Norwegian market for consumers who want to fix their debt interest rate.

In table (3) the approximations are compared for the base case parameters. The refined Cambell-Viceira approximation (10) performs best. The Campbell-Viceira approximation (9) also performs better than the first order condition approximation (11). Some preliminary testing indicates that this ranking holds for maturities up to 5 years. Then the ranking is reversed, i.e., the first order condition performs better than the Campbell-Viceira approximation which again performs better than the refined Cambell-Viceira approximation. When

Base case parameters:	
Time horizon	$T - s = 3$
Amount of debt	$L = 1$
Wealth to debt ratio	$D = 1$
Fixed interest rate	$r^x = 5\%$
Expectation of $\tilde{R}$	$\mu_{s,T} = 0.147079$
Variance of $\tilde{R}$	$\sigma^2_{s,T} = 0.002604$

Table 2: Base case parameters. Observe that the assumed $\mu_{s,T}$ and $\sigma^2_{s,T}$ is consisten with a short term interest process of the Ornstein-Uhlenbeck type $dr_t = q(m - r_t)dt + vdB_t$ with parameters $v = 0.02$, $q = 0.15$, $m = 0.045$, and initial value $r_0 = 0.05$.

	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = 2$	$\rho = 4$	$\rho = 8$
β^*	1.231	0.6187	0.3100	0.1551	0.07759
$\hat{\beta}_{CV}$	1.243	0.6217	0.3109	0.1554	0.07777
$\hat{\beta}_{CV+}$	1.232	0.6180	0.3095	0.1549	0.07748
$\hat{\beta}$	1.249	0.6236	0.3116	0.1557	0.07785
$E[U(W_T^*)]$	2.157	0.1505	-0.8605	-0.2125	-0.04997

Table 3: Optimal β from equation (8), approximate β from equations (9), (10), and (11) and optimal expected utility for the base case parameters.

Parameter	Lower/upper bound $\mu^{\Delta} + \frac{1}{2}\sigma^2_{s,T} = 0$	Numerical result
Δ_x	-5.40 bp	$\Delta_x \uparrow \rightarrow \beta^* \uparrow$
$T - s$	9.38	$T - s \uparrow \rightarrow \beta^* \downarrow$
Δ_m	-22.29 bp	$\Delta_m \uparrow \rightarrow \beta^* \downarrow$
v	0.02996	$v \uparrow \rightarrow \beta^* \downarrow$

Table 4: Comparative statistics from figure 1, 2, 3, and 4.

we present numerical results below we therefore (except for two cases) apply the refined Campbell-Viceira approximation.

In table (3) some numerical values of β^* are calculated together with the values of the approximated β for the ase case parameters.

Figure 1,2,3, and 4 depict the approximated optimal fraction of floating rate debt as a function of the relative risk aversion parameter ρ for alternative values of the fixed rate, the time horizon $T - s$, expectation of $\tilde{R}$ (presented as level of mean reversion m in the corresponding short term process), and variance of $\tilde{R}$ (presented as changes in interest rate volatility v), respectively. All graphs are based on the refined Campbell-Viceira approximation (10) except the graphs for $T - s = 6$ and $T - s = 9$ in Figure 2, which are based and the first order condition approximation (11). The comparative statistics read from the 5 figures are presented in the following table. The arguments after Proposition 1 allow us to find a lower (upper) bound for the parameter values in order for the optimal β to take positive values in the case where β is increasing (decreasing) with the parameter. We sometimes use the standard convention that 1% = 100 bp (basispoints).

4 The dynamic problem: Continuous rebalancing of debt

In this section we allow the investor to rebalance the debt portfolio between the initial time s and the time of expiration T. The rebalancing does not impose any cost for the agent and can be done continuously.

Our methodology is based on the martingale formulation by Pliska (1986) and Cox and Huang (1989) as recently extended by Sørensen (1999) and Munk and Sørensen (2001).

4.1 The bond market

The dynamics of market prices of default free unit discount bonds under the original probability measure P are

$$P_{t,\tau} = P_{s,\tau} + \int_s^t [r_u + b(u,\tau)]P_{u,\tau}du + \int_s^t a(u,\tau)P_{u,\tau}dB_u, \qquad (12)$$

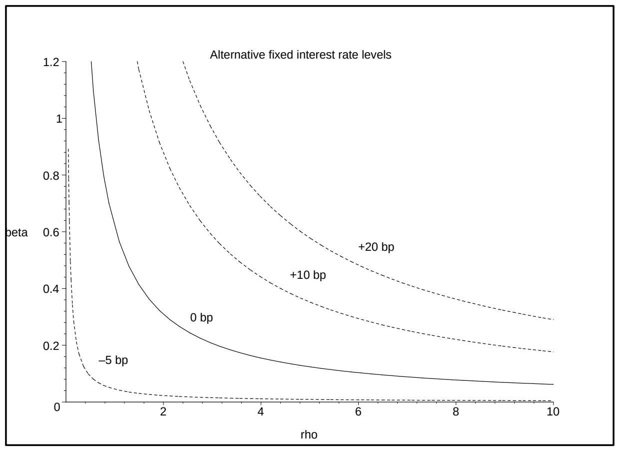

Figure 1: Optimal β as a function of the relativ risk aversion parameter ρ for alternative fixed interest rates r^x. The values of r^x are, starting from the left curve, 4.95%, 5% , 5.1%, and 5.2%. The solid line corresponds to the base case parameter $r^x = 5\%$.

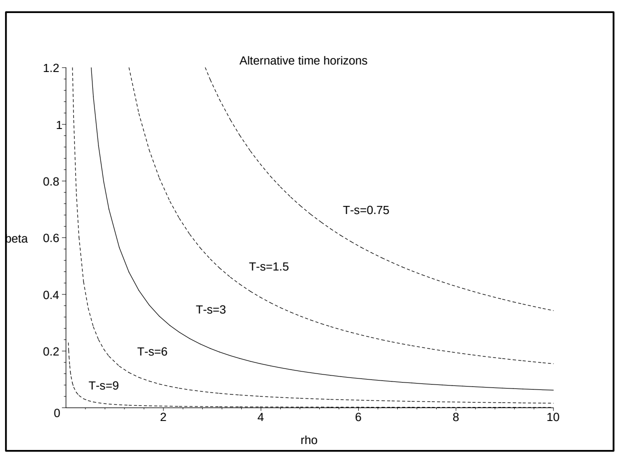

Figure 2: Optimal β as a function of the relativ risk aversion parameter ρ for various *ceteris paribus* changes in the base case parameters. The following values of time horizons $T - s$ have been used, starting from the leftmost curve: 9, 6, 3, $1\frac{1}{2}$, and $\frac{3}{4}$. The solid line corresponds to the base case parameter.

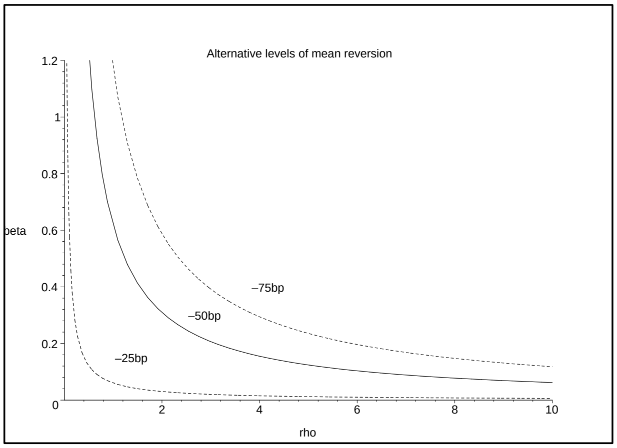

Figure 3: Optimal β as a function of the relativ risk aversion parameter ρ for various *ceteris paribus* changes in the base case parameters. The following values of $\mu_{s,T}$ have been used: 0.148540, 0.147079, and 0.145619. This corresponds to the following changes in the parameter representing the level of mean reversion m: 4.75%, 4.5%, and 4.25% bp. The solid line corresponds to the base case parameter $\mu_{s,T} = 0.147079$ ($m = 4.5\%$).

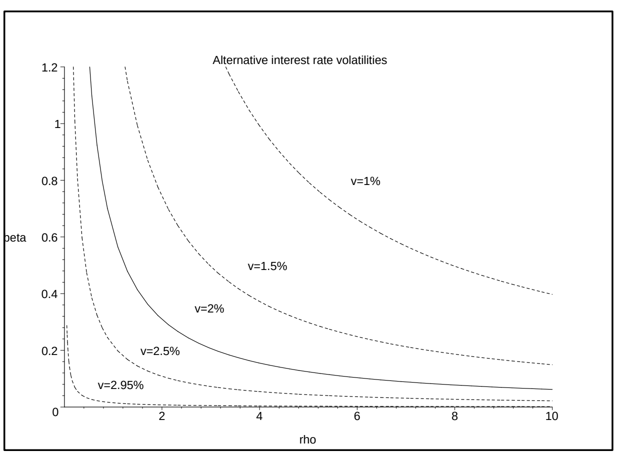

Figure 4: Optimal β as a function of the relativ risk aversion parameter ρ for various *ceteris paribus* changes in the base case parameters. The following values of $\sigma_{s,T}$ have been used: 0.005665, 0.004069, 0.002604, 0.001465, and 0.000651. This corresponds to the following changes in the volatility parameter v of the short term interest rate process: 0.0295, 0.025, 0.02, 0.015, and 0.01. The solid line corresponds to the base case parameter $\sigma_{s,T} = 0.002604$ ($v = 0.02$).

where τ represents the maturity of the bond, and

$$b(t,\tau) = a(t,\tau)\lambda_s(t).$$

Here $\lambda_s(t)$ represents the market price of interest rate risk at time t assesed by the market at the initial time s. We assume that $\lambda_s(t)$ is deterministic. Furthermore, the function $a(t,\tau)$ is determined by the so-far unspecified short interest rate process.

For simplicity we also assume that $\int_s^t r_u du$ for all $t \leq T$ is Gaussian and let

$$\mu_{s,t} = E\left[\int_s^t r_u du\right]$$

and

$$\sigma_{s,t}^2 = \text{Var}\left[\int_s^t r_u du\right].$$

4.2 Intermediate market value of debt and debt dynamics

In the dynamic setting of this section we both need the market value at intermediate points in time as well as the stochastic dynamics of fixed and floating rate debt.

First we derive expressions for the market values of debt at time t, $s \leq t < T$. Let D_s^L be a time s amount of floating rate debt which has to be paid back including interest rates at time T. The market value of this debt at time $t > s$ is

$$\begin{aligned} D_t^L &= D_s^L E_t^Q\left[e^{-\int_t^T r_u du} e^{\int_s^T r_u du}\right] \\ &= D_s^L e^{\int_s^t r_u du}. \end{aligned} \tag{13}$$

Let D_s^X denote a time s amount of fixed rate debt which has to be paid back including interest rates at time T . The market value of this debt at time $t > s$ is

$$\begin{aligned} D_t^X &= D_s^X E_t^Q\left[e^{-\int_t^T r_u du} e^{\int_s^T r_s^x du}\right] \\ &= D_s^X e^{r_s^x (T-s)} E_t^Q\left[e^{-\int_t^T r_u du}\right] \\ &= D_s^X e^{r_s^x (T-s)} P_{t,T}. \end{aligned} \tag{14}$$

From the previous expressions (13) and (14) we derive the fixed and floating rate dynamics below. The market value of fixed rate debt can be described by the following stochastic differential equation

$$dD_t^X = (r_t + b_{t,T}) D_t^X dt + a_{t,T} D_t^X dB_t, \tag{15}$$

together with the given constant D_s^X. As one should expect, fixed rate debt has identical dynamics as a bond with the same expiration as the debt, see expression (12).

The corresponding stochastic differential equation for the market value of floating rate debt is

$$dD_t^L = r_t D_t^L dt, \tag{16}$$

where the initial value is the given constant D_s^L. Floating rate debt has the same dynamics as a bank account (sometimes called a money market account) where interest accrues according to the spot rate r_t.

In addition to debt the agent's time T wealth consists only of the deterministic amount $\bar{W}$. The market value at time t, $s \leq t \leq T$ of the agent's time T wealth for given amounts of fixed and floating rate debts is therefore

$$W_t = \bar{W} P_{t,T} - D_t^X - D_t^L.$$

From the equations (12), (15), (16), and Itô's lemma the dynamics of the wealth process may be written as

$$dW_t = \left((r_t + b_{t,T})W_t + b_{t,T} D_t^L\right) dt + a_{t,T}(W_t + D_t^L) dB_t, \tag{17}$$

where the initial value W_s is a given constant. We now let D_t denote the market value of the total debt at time t, i.e., $D_t = D_t^X + D_t^L$, and let α_t denote the fraction of floating rate debt, i.e., $D_t^L = \alpha_t D_t$. By substituting in expression (17) we obtain

$$dW_t = ((r_t + b_{t,T})W_t + b_{t,T}\alpha_t D_t)\, dt + a_{t,T}(W_t + \alpha_t D_t) dB_t. \tag{18}$$

Alternatively, we may express the floating rate amount at time t as a fraction β_t of time t wealth, $D_t^L = \beta_t W_t$ (somewhat similar as we did in the previous section above expression (8)). By substituting in expression (17) we now obtain

$$dW_t = (r_t + b_{t,T}(1 + \beta_t))W_t)dt + a_{t,T}(1 + \beta_t) W_t dB_t. \tag{19}$$

The connection between α_t and β_t is

$$\begin{aligned} \alpha_t &= \frac{W_t}{D_t}\beta_t, \\ &= L_t \beta_t, \end{aligned} \tag{20}$$

where L_t is the wealth to debt ratio as previously defined. This is exactly the same relationship between α and β as in the static case.

4.3 The agent's problem

The agent's problem is similar as in section 3. Also in this dynamic set-up utility is derived only from time T wealth. At time s the investors problem is:

$$J_s = \sup_{W_T} E_s\left[\frac{1}{1-\rho}(W_T)^{1-\rho}\right]$$

subject to

$$E_s\left[\xi_{s,T} W_T\right] \leq W_s,$$

where

$$\xi_{s,t} = \exp\left(-\int_s^t r_u du - \int_s^t \lambda_s(u)dB_u - \frac{1}{2}\int_s^t \lambda_s(u)^2 du\right) \tag{21}$$

is sometimes called the *state price deflator*. For the special case $\rho = 1$ we assume that $J_s = \sup_{W_T} E_s[\ln(W_T)]$.

For future use define

$$\Lambda_t = \int_t^T \lambda_s(u)^2 du$$

and

$$\mu_t^\Delta = \mu_{t,T} - r^x(T-t).$$

For example the market price of a default free unit discount bond expiring at time T may be expressed by the state price deflator as

$$P_{s,T} = E_s[\xi_{s,T}].$$

Our assumptions on $\int_s^T r_u du$ and $\lambda_s(t)$ imply that $\ln(\xi_{s,T})$ is normally distributed and we can calculate $P_{s,T}$ as

$$P_{s,T} = \exp\left(-\mu_{s,T} - \frac{1}{2}\Lambda_s(u) + \frac{1}{2}V_{s,T}^2\right),$$

where

$$V_{s,T}^2 = \operatorname{Var}\left(\int_s^T r_u du + \int_s^T \lambda_s(u)dB_u | \mathcal{F}_s\right).$$

4.4 Solution of the problem

First define F_s as the forward (time T) value of time s wealth, i.e.,

$$F_s = \frac{W_s}{P_{s,T}} = \frac{LD}{P_{s,T}}.$$

The optimal indirect utility for this problem is given in the following proposition.

Proposition 4 *The optimal expected utility for this problem for $\rho \neq 1$ is*

$$J_s = \frac{1}{1-\rho}\left[F_s^{1-\rho}\exp\left(\frac{1-\rho}{\rho}\left(\frac{1}{2}\Lambda_s + \mu_s^\Delta\right)\right)\right]. \tag{22}$$

Optimal expected utility for logarithmic utility ($\rho = 1$) is

$$J_s = \ln(F_s) + \frac{1}{2}\Lambda_s + \mu_s^\Delta. \tag{23}$$

See appendix C for a proof.

4.5 Optimal debt positions

In the following propositions we present the optimal fractions of floating rate debt and, thus, implicitly, the optimal fixed rate debt.

Proposition 5 *The optimal time $t \geq s$ fraction of floating rate debt, expressed as a fraction of total debt, is*

$$\alpha_t = \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right) L_t.$$

Proposition 6 *The optimal time $t \geq s$ fraction of floating rate debt, expressed as a fraction of the market value of total time t wealth, is*

$$\beta_t = \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right).$$

See appendix C for a proof.

4.6 Numerical illustrations — dynamic case

We assume that the spot interest rate process by r_t is given by the following stochastic differential equation under the original probability measure P

$$dr_t = q(m - r_t)dt + vdB_t, \tag{24}$$

where the initial value $r_s = r$ is a given constant. This is the well known Ornstein-Uhlenbeck process, first used in financial economics by Vasicek (1977). The parameters m, q and v are interpreted as the long-run mean to which the process tend to revert, the speed of reversion and the volatility of the process, respectively.

The solution of (24) is

$$r_t = m + (r_s - m)e^{-q(t-s)} + \int_s^t ve^{-q(t-u)}dB_u. \tag{25}$$

Since $R_{s,T} = \int_s^T r_t dt$ we calculate $R_{s,T}$ as as

$$R_{s,T} = m(T - s) + (r_s - m)\frac{1 - e^{-q(T-s)}}{q} - \int_s^T a(u, T)dB_u, \tag{26}$$

where

$$a(t, \tau) = \frac{v}{q}(e^{-q(\tau-t)} - 1). \tag{27}$$

Observe that $R_{s,T}$ is Gaussian with expectation and variance

$$\mu_{s,T} = m(T - s) + \frac{1}{q}(r_s - m)(1 - e^{-q(T-s)}) \tag{28}$$

and

$$\sigma_{s,T}^2 = \int_s^T a(u,T)^2 du = \frac{v^2}{2q^3}\left(2q(T-s) - 3 + 4e^{-q(T-s)} - e^{-2q(T-s)}\right), \quad (29)$$

respectively.

We assume that the market price of (interest rate) risk at time s as a function of time t is

$$\lambda_s(t) = \frac{qm}{v} - \frac{1}{v}\left[q f_s(t) + \frac{\partial}{\partial t} f_s(t)\right] - \frac{v}{2q}(1 - e^{-2q(t-s)}). \quad (30)$$

Notice that for fixed s the market price of risk is a deterministic process of time which depends on the time s forward rates $f_s(t)$ as well as the derivative of the time s forward rate $\frac{\partial}{\partial t} f_s(t)$ and 3 parameters (q, m, v) of the spot interest rate process.

By this choice of market price of risk the dynamics of the spot interest rate process under the equivalent martingale measure Q can be written as

$$dr_t = q\left(\theta_t - r_t\right) dt + v d\hat{B}_t$$

where $\hat{B}_t$ is a Brownian motion under the equivalent martingale measure Q, $r_s = r$ a constant, and

$$\theta_t = \frac{1}{q}\frac{\partial}{\partial t} f_s(t) + f_s(t) + \frac{v^2}{2q^2}(1 - e^{-2q(t-s)}).$$

This model[3] of spot interest rates under the equivalent martingale measure is known as a version of the Hull and White (1990) one-factor model.

We present some numerical results to compare the dynamic case with the static case both in terms of welfare measured by expected utility and initial fractions of floating rate debt.

From proposition 5 the optimal fraction of floating rate debt depends on the market price of risk, which again (see equation (30)) depends on the initial (time s) forward rates. We will therefore consider the following 4 cases (see Figure 6):

[3] Alternatively, our model can be expressed under the equivalent martingale measure as

$$r_t = f_s(t) + \int_s^t q(\hat{\theta}_u - r_u) du + \int_s^t v d\hat{B}_u,$$

where

$$\hat{\theta}_t = f_s(t) + \frac{v^2}{2q^2}(1 - e^{-2q(t-s)}).$$

This latter equivalent formulation is in spirit with the more general Heath, Jarrow, and Morton (1992) term structure formulation in the sense that the initial value of the process is the instantaneous forward rate. It can also be shown that the current model is a special case of the Heath et al. (1992) model, see e.g., Miltersen and Persson (1999). The above equation (30) corrects an error in corresponding equation (for λ_t) at the bottom of page 310 in Miltersen and Persson (1999).

Base case parameters:	
Speed of mean reversion	$q = 0.15$
Diff. in mean reversion level (1% = 100 bp)	$r - m = \Delta_m = -50$ bp
Volatility of interest rate	$v = 0.02$
Other parameters:	
Time horizon	$T - s = 3$
Difference in fixed interest rate level	$r - r^x = \Delta_x = 0$
Additional parameters for analyzing welfare:	
Amount of debt	$L = 1$
Wealth to debt ratio	$D = 1$
Interest rate level	$r = 5\%$

Table 5: Base case parameters

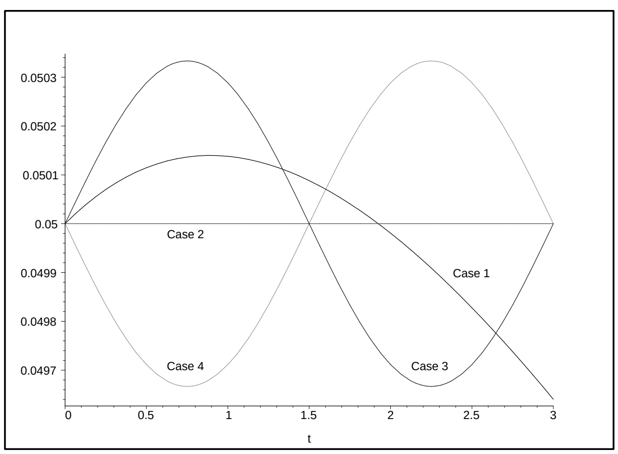

Figure 5: The 4 different initial term structures.

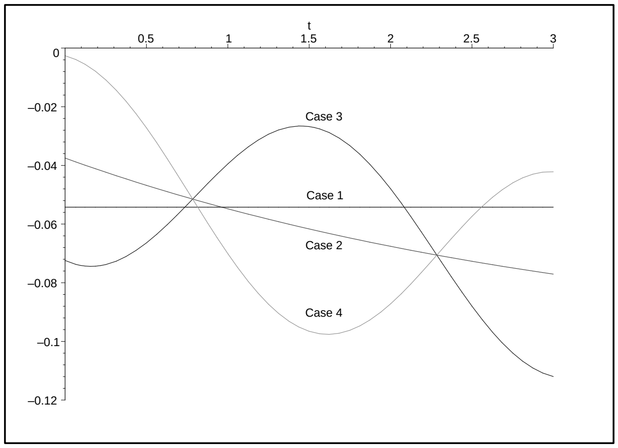

Figure 6: The 4 different market price of risk processes.

- Case 1. As Brennan and Xia (2000) and Bajeux-Besnainou et al. (2001) among others, we first assume that $\lambda_s(t) = \bar{\lambda}$, a constant. This assumption implies (from equation (30)) that the initial forward rates are of the form:

$$f_s^{(1)}(t) = re^{-q(t-s)} + (m - \frac{v\bar{\lambda}}{q})(1 - e^{-q(t-s)}) - \frac{v^2}{2q^2}(1 - e^{-q(t-s)})^2.$$

The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f_s^{(1)}(t) = qe^{-q(t-s)} \left(m - \frac{v\bar{\lambda}}{q} - r - \frac{v^2}{q^2}(1 - e^{-q(t-s)}) \right).$$

For our choice of parameter values $f_s^{(1)}(t)$ will be humped, i.e., increasing for small t values and decreasing for larger t values.

By definition $P_{s,T} = e^{r^x(T-s)}$. Also, $P_{s,T}$ can be calculated as

$$P_{s,T} = E_s^Q[e^{-\int_s^T r_t dt}] = e^{-\hat{\mu}_{s,T} + \frac{1}{2}\sigma_{s,T}^2} = e^{-\mu_{s,T} + \frac{1}{2}\sigma_{s,T}^2 + \bar{\lambda}\frac{v}{q}[T - s - \frac{1}{q}(1 - e^{-q(T-s)})]},$$

where $\hat{\mu}_{s,T}$ denotes the expectation of $\int_s^T r_t dt$ under the equivalent martingale measure. By equating these two expressions the market price of risk at time s is determined as

$$\bar{\lambda} = \frac{q}{v} \frac{\mu_{s,T} - \frac{1}{2}\sigma_{s,T}^2 - r^x(T-s)}{(T - s - \frac{1}{q}(1 - e^{-q(T-s)}))}.$$

- Case 2. The initial forward rates are constant, i.e., $f_s^{(2)}(t) = r$ for all t. Then $\frac{\partial}{\partial t} f_s^{(2)}(t) = 0$ and

$$\lambda_s^{(2)}(t) = \frac{q}{v}(m-r) - \frac{v}{2q}(1 - e^{-2q(t-s)}).$$

- Case 3. The initial forward rates are initially increasing, given by the function

$$f_s^{(3)}(t) = r + \sin(\frac{2\pi(t-s)}{T-s})\frac{1}{3000}.$$

The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f_s^{(3)}(t) = \cos(\frac{2\pi(t-s)}{T-s})\frac{\pi}{1500(T-s)}$$

and

$$\lambda_s^{(3)}(t) = \frac{q}{v}(m-r) - \frac{q}{3000v}\sin(\frac{2\pi(t-s)}{T-s}) - \frac{\pi}{1500(T-s)v}\cos(\frac{2\pi(t-s)}{T-s}) - \frac{v}{2q}(1 - e^{-2q(t-s)}).$$

- Case 4. The forward rates are initially decreasing. In particular the initial forward rates are given by the function

$$f_s^{(4)}(t) = r + \sin(\frac{2\pi(t-s)}{T-s} + \pi)\frac{1}{3000}.$$

The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f_s^{(4)}(t) = \cos(\frac{2\pi(t-s)}{T-s} + \pi)\frac{\pi}{1500(T-s)}$$

and

$$\lambda_s^{(4)}(t) = \frac{q}{v}(m-r) - \frac{q}{3000v}\sin(\frac{2\pi(t-s)}{T-s} + \pi) - \frac{\pi}{1500(T-s)v}\cos(\frac{2\pi(t-s)}{T-s} + \pi) - \frac{v}{2q}(1 - e^{-2q(t-s)}).$$

These choices of initial forward rates all produce the same fixed rate $r^x = 5\%$. To make the choice between fixed and floating rate less obvious we set the initial spot rate equal to the fixed rate in the numerical examples. Given this restriction the sinus function is a natural choice as a model of the initial forward rates in case 3 and case 4.

In table (6) some values of J_s are calculated for the four cases for the base case parameters. In order to interpret these results the increase in *certainty equivalent wealth* from the static case to each of the dynamic case is presented

J_s	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = 2$	$\rho = 4$	$\rho = 8$
static case	2.157	0.1505	-0.8605	-0.2125	-0.04997
constant λ	2.159	0.1515	-0.8601	-0.2123	-0.04992
constant $f_s(t)$	2.162	0.1527	-0.8595	-0.2121	-0.04987
increasing $f_s(t)$	2.163	0.1535	-0.8592	-0.2120	-0.04984
decreasing $f_s(t)$	2.164	0.1538	-0.8591	-0.2119	-0.04983

Table 6: Optimal initial utility levels J_s calculated from equation (22) and compared with the results of the previous static model in table (**??**) for the base case parameters.

ΔCE in %	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = 2$	$\rho = 4$	$\rho = 8$
constant λ	0.19	0.10	0.05	0.03	0.01
constant $f_s(t)$	0.46	0.22	0.12	0.06	0.03
increasing $f_s(t)$	0.56	0.30	0.15	0.08	0.04
decreasing $f_s(t)$	0.65	0.33	0.16	0.09	0.04

Table 7: Percentage increase in certainty equivalent wealth (ΔCE) compared with static case for the four dynamic cases. Let $\bar{u}$ denote the optimal utility level from table (6). The certainty equivalent wealth is then calculated as $(\bar{u}(1-\rho))^{\frac{1}{1-\rho}}$ for $\rho \neq 1$ and as $e^{\bar{u}}$ for $\rho = 1$.

in table (7). The overall conclusion is that the increase in optimal expected utility from the static to the dynamic case is low, less that 0.65%. The increase in certainty equivalent wealth seems to be decreasing in the risk aversion parameter ρ (with one exception), so that the welfare increase in a dynamic setting is largest for investors with low levels of risk aversion.

In the figures 8-13 the optimal values for the initial fractions of floating rate debt β_s are presented. The main conclusion from these tables is that the initial positions of floating rate are sensitive to the initial term structure.

5 Concluding remarks and further research

Our model for debt allocation produces realistic debt allocations for reasonable parameter values, and can be applied as one tool in individuals' financing decisions. If we interpret the optimizing agent as a representative agent our model can be used for analyzing the distribution between fixed and adjustable rates in mortgage markets.

Our preliminary numerical comparisons between the static case (no rebalancing of the debt portfolio) with the dynamic case (continuous and costless rebalancing) indicate, perhaps surprisingly, low increase in 'welfare' in dynamic situation compared to static situation. At least this is the case for high levels of relative risk aversion.

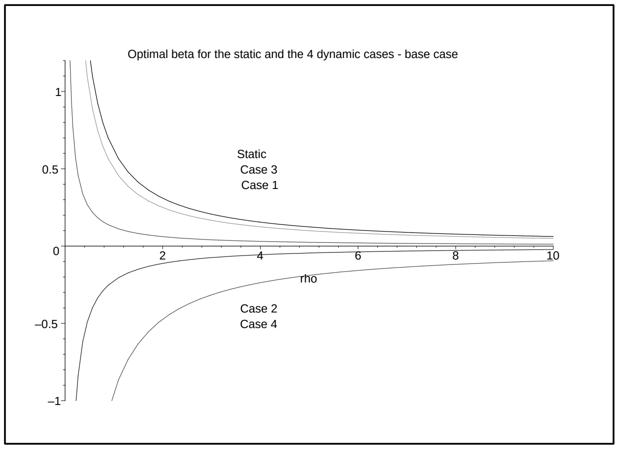

Figure 7: Initial optimal allocations for the four dynamic cases and the static case for the base case parameters.

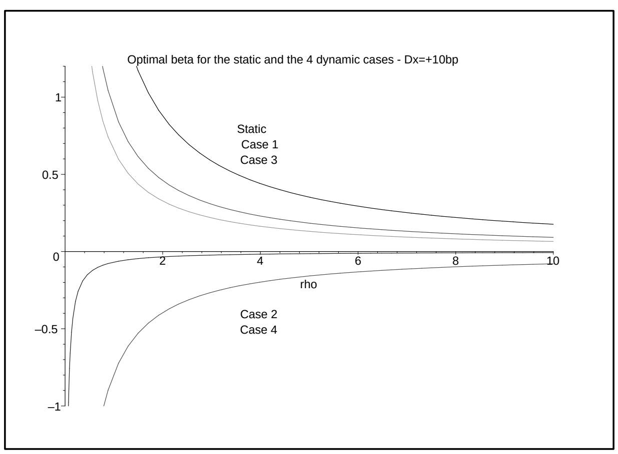

Figure 8: Initial optimal allocations for the four dynamic cases and the static case for $\Delta_x = +10$ bp.

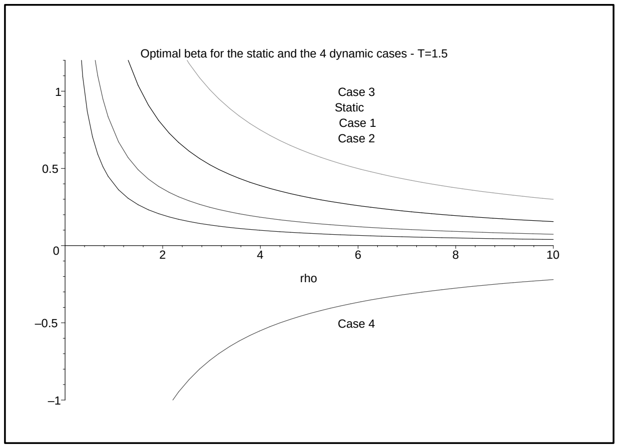

Figure 9: Initial optimal allocations for the four dynamic cases and the static case for $T = 1.5$.

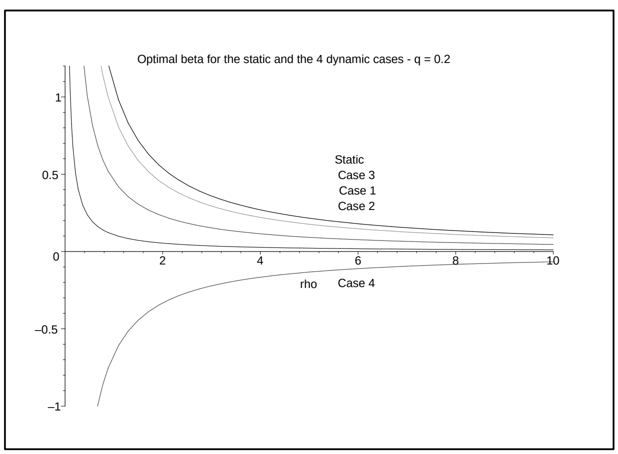

Figure 10: Initial optimal allocations for the four dynamic cases and the static case for $q = 0.2$.

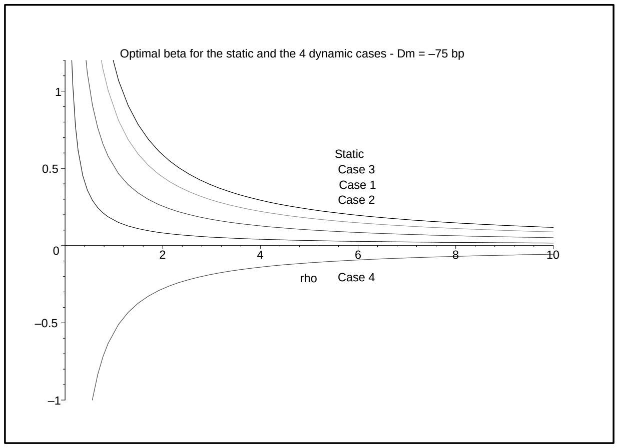

Figure 11: Initial optimal allocations for the four dynamic cases and the static case for $\Delta_m = -75$ bp.

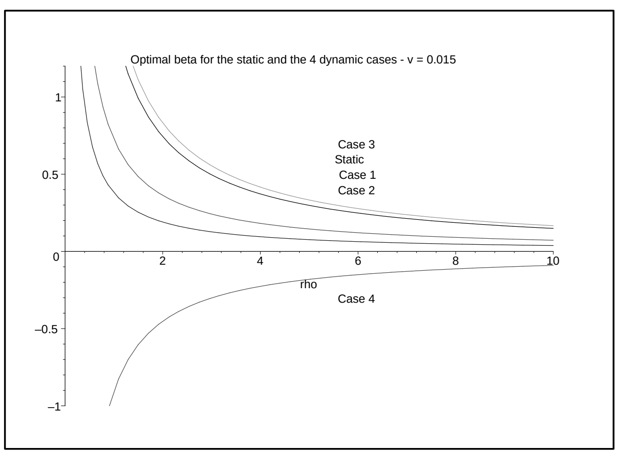

Figure 12: Initial optimal allocations for the four dynamic cases and the static case for $v = 0.015$.

In the dynamic case the optimal initial fractions of floating rate debt are partly determined by the initial forward interest rates, which do not influence the corresponding optimal fraction in the static case. Therefore, we do not learn anything about the optimal floating rate debt fraction in the dynamic case from the static case. Even if we are willing to assume that the market price of interest rate risk is constant the initial optimal fractions of floating rate debt are different in the static and dynamic cases.

The research in this article can be extended in a number of ways. First, realism can be improved by introducing a multi-factor interest rate model. Also, transaction costs can be introduced, in the spirit of Davis and Norman (1990), Korn (1998), Øksendal and Sulem (2000), and Zakamouline (2002) in order to make the set-up closer to real world situations. Finally, this set-up may be used to study the effect of a stochastic collateral ($\bar{W}$).

A Appendix

In this appendix a number of detailed calculations is collected.

From equation (30) direct calculations give

$$\int_s^T \lambda_s(t)dt = \frac{q}{v}\left[(m - r^x)(T-s) + \frac{1}{q}(r - f_s(T))\right] - \frac{q}{v}\left[\frac{v^2}{4q^3}(2q(T-s) - 1 + e^{-2q(T-s)})\right] \quad (31)$$

and

$$\int_s^T \lambda_s(t)e^{qt}dt = \frac{q}{v}e^{qT}\left[\frac{m}{q}(1 - e^{-q(T-s)}) - \frac{1}{q}(f_s(T) - re^{-q(T-s)})\right] + \frac{q}{v}e^{qT}\left[\frac{v^2}{4q^3}(4e^{-q(T-s)} - 2e^{-2q(T-s)} - 2)\right]. \quad (32)$$

From the equations (30) and (26) by using (31) and (32) it follows that

$$\begin{aligned}\mathrm{Cov}\left(\int_s^T r_t dt, \int_s^T \lambda_s(t)dB_t|\mathcal{F}_s\right) &= \int_s^T -a(t,T)\lambda_s(t)dt \\ &= \frac{v}{q}\left[\int_s^T \lambda_s(t)dt - e^{qT}\int_s^T \lambda_s(t)e^{qt}dt\right] \\ &= \mu^\Delta - \frac{1}{2}\sigma^2_{s,T}.\end{aligned}$$

It is now straight forward to calculate

$$
\begin{aligned}
V_{s,T}^2 &= \mathrm{Var}\left(\int_s^T r_t dt + \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right) \\
&= \sigma_{s,T}^2 + \int_s^T \lambda_s(t)^2 dt + 2\mathrm{Cov}\left(\int_s^T r_t dt, \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right) \\
&= \int_s^T \lambda_s(t)^2 dt + 2\mu^{\Delta}. \qquad (33)
\end{aligned}
$$

This expression also holds in the case where $\lambda_s(t) = \bar{\lambda}$, a constant.

B An approximation of β^*

Define

$$X = e^Y,$$

where $Y \sim N(\mu^{\Delta}, \sigma_{s,T}^2)$. Moreover, let

$$f(X) = (1 + \beta(1 - X))^{-\rho}, \qquad g(X) = f(X)X.$$

Then the first order condition (8) can be written as

$$E[f(X)] - E[g(X)] = 0. \qquad (34)$$

We approximate $f(\cdot)$ and $g(\cdot)$ by second order Taylor expansions. First we calculate derivatives:

$$
\begin{aligned}
f'(X) &= \rho\beta(1 + \beta(1 - X))^{-(\rho+1)}, & f''(X) &= \rho(\rho + 1)\beta^2(1 + \beta(1 - X))^{-(\rho+2)}, \\
g'(X) &= f'(X)X + f(X), & g''(X) &= f''(X)X + 2f'(X).
\end{aligned}
$$

The first order condition (34) based on second order Taylor approximations of $f(\cdot)$ and $g(\cdot)$ around $X = E[X] = \mu$ now becomes

$$f(\mu)(1 - \mu) - \sigma^2 f'(\mu) + \frac{1}{2}\sigma^2 f''(\mu)(1 - \mu) = 0,$$

where $\sigma^2 = \mathrm{Var}(X)$. Inserting for $f(\cdot)$, $f'(\cdot)$,$f''(\cdot)$, simplifying, and rearranging yields

$$((1 - \mu)^2 + \frac{1}{2}\sigma^2\rho(\rho - 1))\beta^2 + (2 - \frac{\sigma^2\rho}{1 - \mu})\beta + 1 = 0.$$

This equation has the solution

$$\beta = \frac{1}{A}\left(-\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - A}\right), \qquad (35)$$

where

$$A = (1-\mu)^2 + \frac{1}{2}\sigma^2\rho(\rho-1),$$

and

$$B = 2(1-\mu) - \frac{\sigma^2\rho}{1-\mu}.$$

The approximation, labelled $\hat{\beta}$, in expression (11) is calculated as the solution of (35) using the negative root.

C Proofs

C.1 Proof of proposition 4

Consider first the case $\rho \neq 1$. From the first order condition of the corresponding Lagrangian we obtain

$$W_T = \mathcal{L}^{-\frac{1}{\rho}}(\xi_{s,T})^{-\frac{1}{\rho}}, \tag{36}$$

where $\mathcal{L}$ denotes the Lagrangian multiplier. Inserting the expression (36) for W_T into the budget constraint we obtain

$$W_s = \mathcal{L}^{-\frac{1}{\rho}} E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}],$$

from which we determine $\mathcal{L}$ as $\mathcal{L}^{-\frac{1}{\rho}} = \frac{W_t}{E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]}$. From equation (36) we write the optimal terminal wealth W_T^* as

$$W_T^* = \frac{W_s}{E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]}(\xi_{s,T})^{-\frac{1}{\rho}}. \tag{37}$$

Finally, we insert this expression into the objective function and obtain

$$J_s = E_s\left[\frac{1}{1-\rho}(W_T^*)^{1-\rho}\right] = \frac{1}{1-\rho} W_s^{1-\rho} E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]^{\rho}.$$

Equation (22) is obtained by calculating $E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}] = (P_{s,T})^{\frac{\rho-1}{\rho}} e^{\frac{1}{2}\frac{1-\rho}{\rho^2}V_{s,T}^2}$.

Equation (36) also holds for the case $\rho = 1$. The expression corresponding to equation (37) is $W_T^* = W_s \frac{1}{\xi_{s,T}}$. Equation (23) follows by inserting this expression into the objective function.

C.2 Proof of proposition 5

The proof consists of deriving the dynamics of the optimal wealth process (37). By equating this process with the wealth processes derived earlier in equations (19) and (18) the optimal β and α, respectively, are determined.

We start by defining the process Y_t for $t \geq s$ as

$$Y_t = \frac{W_s}{Q_{s,t}}(\xi_{s,t})^{-\frac{1}{\rho}}, \tag{38}$$

where

$$Q_{s,t} = E_s\left[(\xi_{s,t})^{\frac{\rho-1}{\rho}}\right] = (P_{s,t})^{\frac{\rho-1}{\rho}}\, e^{\frac{1}{2}\frac{1-\rho}{\rho^2}V_{s,t}^2}.$$

Observe that Y_t can be interpreted as the optimal wealth process for the given time horizon t, in particular $Y_T = W_T^*$ from equation (37). By applying Itô's lemma to the above equation $Q_{s,t}$ for $t \geq s$ may be expressed as

$$Q_{s,t} = 1 + \int_s^t (\cdot)\, dv + \int_s^t \frac{\rho-1}{\rho} a_{v,t} Q_{s,v} dB_v, \tag{39}$$

where the drift term is left unspecified. Furthermore, we obtain from equation (21)

$$\xi_{s,t} = 1 - \int_s^t r_v \xi_{s,v} dv - \int_s^t \lambda_s(v) \xi_{s,v} dB_v. \tag{40}$$

We now apply Itô's lemma to equation (38) to find the dynamics of Y_t and evaluate this expression for $t = T$:

$$W_T^* = W_s + \int_s^T (\cdot)\, dv + \int_s^T \left[\frac{\rho-1}{\rho} a_{v,T} + \frac{\lambda_s(v)}{\rho}\right] W_v dB_v$$

The time t instantaneous dB_t term of this equation is $\left[\frac{\rho-1}{\rho} a_{t,T} + \frac{\lambda_s(t)}{\rho}\right] W_t$. By equating this term of with the similar term $a_{t,T}(1+\beta_t)W_t$ of expression (19) the expression for β in the proposition is obtained.

Proposition (5) then follows from the general connection between α_t and β_t in expression (20). Alternatively, it can be derived by equating the dB_t term of the above equation with the dB_t term of equation (18).

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